Growth and bubbles: The interplay between productive investment and the cost of rearing children

Preliminary version

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Abstract

We study the role of bubble on growth when agents choose to invest in a productive asset (human capital, project) and decide their number of children. The bubble is used to smooth consumption over the lifecycle, but can also be used to finance the productive investment. We show and explain that the time cost of rearing children plays a key role in the analysis. If the time cost per child is sufficiently large, households have only a few number of children. The bubble has a crowding-in effect because it is used to finance the productive investment. On the contrary, if the time cost per child is low enough, households have several children. Then, the bubble is in particular used to finance the global cost of rearing children and has a crowding-out effect. Therefore, the new mechanism on the link between growth and bubble we highlight shows that a bubble enhances growth only if the economy is characterized by a high rearing time cost per child, which may be a feature of more developed countries.

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1 Introduction

To be written...

2 Model

Time is discrete \((t = 0, 1, ..., +\infty)\) and there are two types of agents, households and firms.

2.1 Firms

Aggregate output is produced by a continuum of firms, of unit size, using labor, \(L_t\), and capital, \(K_t\), as inputs. As we will see later, capital is interpreted here in a large sense. It encompasses human capital (training, PhD), but also investment in some project (shops, start-up...). In addition, the production benefits from an externality that summarizes a learning by doing process and allows to have endogenous growth. Following Frankel (1962) or Ljungqvist and Sargent (2004), this externality depends on the capital-labor ratio.

Letting \(k_t \equiv K_t/L_t\), \(\bar{k}_t\) represents the average capital-labor ratio. Firms produce the final good with the following technology:

\[
Y_t = F(K_t, \bar{k}_t L_t)
\]

The technology \(F(K_t, \bar{k}_t L_t)\) has the usual neoclassical properties, i.e. is a strictly increasing and concave production function satisfying the Inada conditions, and is homogeneous of degree one with respect to its two arguments.

Profit maximization under perfect competition implies that the wage \(w_t\) and the return of capital \(q_t\) are given by:

\[
w_t = F_2(K_t, \bar{k}_t L_t)\bar{k}_t
\]

\[
q_t = F_1(K_t, \bar{k}_t L_t)
\]

2.2 Households

We consider an overlapping generations economy, where each generation is populated by agents living for three periods. An agent is young at the first period of life, adult at the second period and old at the third period. It corresponds to the periods where an agent is active. As it is argued by Geanokoplos et al. (2004), such a demographic structure is a reasonable representation of the households’ life-cycle.

Each household has utility for consumption at each period of time and for children when she is an adult. Preferences of an individual born in period \(t\) are represented by the following utility function:

\[
\ln c_{1,t} + \beta \ln c_{2,t+1} + \beta^2 \ln (c_{3,t+2} + \epsilon) + \mu \ln n_{t+1}
\]

\(^1\)See Chapter 14.

\(^2\)We note \(F_i(.,.)\) the derivative with respect to the \(i\)th argument of the function.
where \( \beta \in (0, 1) \) is the subjective discount factor, \( \mu \in (0, 1) \) measures the love for children and \( n_{t+1} \) the number of children. At each age, the household can consume the market good produced by firms. Hence, \( c_{j,t} \) amounts for consumption of this good when young \( (j = 1) \), adult \( (j = 2) \) and old \( (j = 3) \). We also assume that when old, the household has a consumption of a home-produced good \( \epsilon > 0 \), which is, to simplify, a perfect substitute of the market good.\(^3\)

As explained by Aiguar and Hurst (2005), Hurst (2008) and Schwerdt (2005), home production at the retirement is quite realistic since it seems to solve the consumption puzzle when households are retired.\(^4\)

While each household uses her time endowment for home production when old, she supplies labor to firms when young and adult. When young, this labor supply is one unit. In contrast, labor supply is endogenous when adult. Indeed, since each household has \( n_{t+1} \) children at the end of the first period, she faces a rearing cost \( \psi w_{t+1} \) per child in terms of the consumption good in middle-age, where \( 0 < \psi < 1 \) measures the fraction of time an adult spends for each child. Having children takes time when adult. Such a specification is often used in the literature on fertility (de la Croix and Doepke (2003), Galor (2005)).

When young, the household invests an amount \( a_{t+1} \) in a productive asset. This investment corresponds to accumulation of human capital, because of training, education or acquisition of new skills (PhD or Executive MBA for instance) or an investment in a project like a shop, a start-up... It provides some adding returns \( q_{t+1} a_{t+1} \) at the period where the agent is working for the market good sector, i.e. when she is adult, but no return when she is retired, i.e. when old.\(^5\) This means that while wage is the only income when young, the return of productive investment is a complementary source of income to wage in the middle-age.

Households may also invest in the speculative asset when young \( (b_{1,t}) \) and adult \( (b_{2,t+1}) \). This asset is supplied in one unit and \( b_{i,t} \) represents the value of the share of this asset bought \( (b_{i,t} > 0) \) or sold \( (b_{i,t} < 0) \) by one young or adult household. Since this asset has no fundamental value, it is a bubble if its price is strictly positive. This last one \( \hat{B}_t \) is given by the sum of the values of the shares held by households living at the same time, \( \hat{B}_t = N_t b_{1,t} + N_{t-1} b_{2,t} \). When \( \hat{B}_t = 0 \) and \( b_{1,t} = b_{2,t} = 0 \), the price of the speculative asset is zero and there is no bubble.

Accordingly, the budget constraints faced by an household born in period \( t \) are:

\[

c_{1,t} + a_{t+1} + b_{1,t} = w_t \tag{4}
\]
\[
c_{2,t+1} + b_{2,t+1} = q_{t+1} a_{t+1} + R_{t+1} b_{1,t} + w_{t+1} \left( 1 - \psi n_{t+1} \right) \tag{5}
\]
\[
c_{3,t+2} = R_{t+2} b_{2,t+1} \tag{6}
\]

\(^3\)For home-produced good, the reader can refer to Gronau (1986) for a survey and Benhabib et al. (1991) for a macro-model.

\(^4\)This puzzle is based on the observation that consumption expenditures are lower when retirements but not consumption of goods.

\(^5\)This form of capital fully depreciates after one period of use.
where $0 < n_{t+1} < 1/\psi$. Population size of the generation born at period $t$ is $N_t$. Therefore, the evolution of the population size of the successive generations is given by $N_{t+1} = n_{t+1}N_t$.

3 The economy without bubble

We first analyze the model without bubble, i.e. $b_{1,t} = b_{2,t} = 0$. It corresponds to our benchmark case and it will allow us to compare the properties of equilibria with and without bubble.

3.1 Household’s choices

Households smooth consumption between young and middle-ages. Maximizing utility (3) under the budget constraints (4)-(6) with $b_{1,t} = b_{2,t+1} = 0$, we obtain the level of productive investment and the number of children:

$$a_{t+1} = \frac{1}{1 + \mu + \beta} \left[ (\mu + \beta)w_t - \frac{w_{t+1}}{q_{t+1}} \right]$$

(7)

$$n_{t+1} = \frac{\mu}{1 + \mu + \beta} \frac{w_t + w_{t+1}/q_{t+1}}{\psi w_{t+1}/q_{t+1}}$$

(8)

On the one hand, one can easily understand why productive investment, which corresponds to saving in the young age, increases with the wage earned when young and decreases with the present value of the wage received when adult. On the other hand, the number of children increases with the lifetime income, $w_t + w_{t+1}/q_{t+1}$, but decreases with the discounted value of the rearing cost of having children $\psi w_{t+1}/q_{t+1}$. As a result, the number of children decreases following an increase of the wage growth factor $w_{t+1}/w_t$.

3.2 Bubbleless BGP

At equilibrium, we have $k_t = \bar{k}_t$. Let us define $\alpha \equiv F_1(1,1)/F(1,1) \in (0,1)$ the capital share in total production and $A \equiv F(1,1) > 0$. Using (1) and (2), we deduce that:

$$w_t = (1 - \alpha)Ak_t$$

(9)

$$q_t = \alpha A$$

(10)

Equilibrium on the labor market requires that:

$$L_t = N_t + N_{t-1}(1 - \psi n_t) = N_t(1 - \psi) + N_{t-1}$$

(11)

Equilibrium on the capital market is satisfied if $N_t a_{t+1} = K_{t+1}$. Using (11), we get:

$$a_{t+1} = \rho(n_{t+1})k_{t+1}$$

(12)
with:

\[ \rho(n_{t+1}) \equiv 1 + (1 - \psi)n_{t+1} \quad (13) \]

Let us note \( \gamma_{t+1} \equiv k_{t+1}/k_t \) the growth factor of the capital-labor ratio. Using (7) and (8), we obtain:

\[ \gamma_{t+1} = \alpha A [\psi(1 + \mu + \beta) + \alpha \mu] = \gamma_{wb} \quad (14) \]

where \( \gamma_{wb} > 0 \) if and only if \( \psi > \alpha \mu / [\mu + \beta(1 - \alpha)] \). We immediately jump on this BGP. Using (8), (9) and (10), we also have:

\[ n_{t+1} = \frac{\mu}{\psi(1 + \mu + \beta)} \left( 1 + \frac{\alpha A}{\gamma_{t+1}} \right) \quad (15) \]

Since the wage increases with the capital-labor ratio and the number of children decreases with the wage growth factor, population growth reduces with the growth factor \( \gamma_{t+1} \). As it is illustrated by Galor (2005), this negative relationship is in accordance with what we observe in Western Europe and in US, Australia, New Zealand and Canada since one century.

Substituting (14) in (15), we obtain the population growth factor at the BGP:

\[ n_{wb} = \frac{\mu}{\psi(1 + \mu + \beta)} \left( 1 + \frac{\alpha A}{\gamma_{t+1}} \right) \quad (16) \]

Note that \( n_{wb} < 1/\psi \) if and only if \( \psi > \psi_{nwb} \), which is larger than \( \psi_{wb} \) and smaller than 1 if \( \mu < \mu_{nwb} \), with:

\[ \psi_{nwb} \equiv \frac{\alpha \mu}{\beta(1 - \alpha)} \quad \text{and} \quad \mu_{nwb} \equiv \frac{\beta(1 - \alpha)}{\alpha} \quad (17) \]

We deduce the following proposition:

**Proposition 1** Without bubble \( b_{1,t} = b_{2,t} = 0 \), the economy immediately jumps on the equilibrium \( \gamma_{t+1} = \gamma_{wb} \) and \( n_{t+1} = n_{wb} \), where \( n_{wb} < 1/\psi \) if and only if \( 1 > \psi > \psi_{nwb} \) and \( \mu < \mu_{nwb} \).

Without bubble, there is no transitional dynamics. This is a standard property of models with endogenous growth and an Ak technology. We also note that there is endogenous growth, \( \gamma_{wb} > 1 \), if the productivity \( A \) is high enough.

## 4 Equilibrium with a bubble

When there is a bubble \( b_{1,t} \neq 0 \) and/or \( b_{2,t} \neq 0 \), an household maximizes her utility (3) under the budget constraints (4)-(6) taking into account that she can also smooth consumption using the speculative asset. By arbitrage, we get
\( R_{t+1} = q_{t+1} \). Then, the number of children and the consumptions over the life-cycle are given by:

\[
\begin{align*}
  n_{t+1} &= \frac{\mu}{1 + \beta + \beta^2 + \mu} \cdot \frac{w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}}}{\psi} \\
  c_{1t} &= \frac{1}{1 + \beta + \beta^2 + \mu} \left( w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}} \right) \\
  c_{2t+1} &= \frac{\beta R_{t+1}}{1 + \beta + \beta^2 + \mu} \left( w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}} \right) \\
  c_{3t+2} &= \frac{\beta^2 R_{t+1}R_{t+2}}{1 + \beta + \beta^2 + \mu} \left( w_t + \frac{w_{t+1}}{R_{t+1}} - \frac{1 + \beta + \mu}{1 + \beta + \beta^2 + \mu} \epsilon \right)
\end{align*}
\]

Since when old, the market good produced by the firms is substitutable to a home-produced good, we should take care that \( c_{3t+2} \) is non-negative. However, as we will see later, the growing disposable income \( w_t + w_{t+1}/R_{t+1} \) ensures a positive consumption \( c_{3t+2} > 0 \) at the long run equilibrium with endogenous growth we are interested in.

Let \( \lambda_t \equiv \epsilon/k_t \). Then, using (4), (6), (9) and (10), we determine the shares of the bubble hold by the young and adult households detrended by the capital-labor ratio:

\[
\begin{align*}
  \frac{b_{1t}}{k_t} &= (1 - \alpha)A \frac{\beta + \beta^2 + \mu}{1 + \beta + \beta^2 + \mu} \\
  b_{2t} &= \beta^2 (1 - \alpha)A \frac{\alpha A}{1 + \beta + \beta^2 + \mu} \left( \frac{\alpha A}{\gamma_{t+1}} + 1 \right) - \frac{\lambda_t}{\alpha A} \frac{1 + \beta + \mu}{1 + \beta + \beta^2 + \mu} \equiv \tilde{b}_2(\gamma_t, \lambda_t) \\
  n(\gamma_{t+1}, \lambda_{t+1}) &\equiv \frac{\mu}{\psi} \frac{\alpha A}{\gamma_{t+1}} \left( 1 + \frac{\lambda_{t+1}}{\alpha(1 - \alpha)A^2} \right) = n_{t+1}
\end{align*}
\]

By inspection of equation (22), we see that \( \tilde{b}_1(\gamma_{t+1}, \lambda_{t+1}) \) decreases with the growth factor \( \gamma_{t+1} \). Indeed, using the budget constraint (4), we observe that the share of the bubble bought by a young household decreases with productive investment and consumption at the first period of life. The first effect is summarized by \( \gamma_{t+1} \rho(\gamma_{t+1}, \lambda_{t+1}) \), which is of course increasing in \( \gamma_{t+1} \), while the second one is explained by the fact that the consumption when young linearly raises with the life-cycle income (see equation (19)). Using (23), we note that \( \tilde{b}_2(\gamma_t, \lambda_t) \) is decreasing in \( \gamma_t \). Indeed, since the share of the bubble hold in middle-age is the saving when adult, it increases with the disposable income. This explains that the share of the bubble hold in the middle-age detrended by the capital-labor ratio decreases with the growth factor. Finally, using equation (24), we note that the number of children decreases with the growth factor. As we explain in the economy without bubble, such a relationship is observed in
many developed countries since the beginning of the last century (Galor (2005)). The explanation of this link between $n_{t+1}$ and $\gamma_{t+1}$ is also the same than in the previous section. The number of children raises with the life-cycle income, but decreases with the discounted value of the rearing cost $\psi w_{t+1}/R_{t+1}$.

Equilibrium on the speculative asset market means that:

$$N_{t+1}b_{t+1} + N_{t}b_{2t+1} = R_{t+1}(N_{t}b_{1t} + N_{t-1}b_{2t}) \quad (25)$$

where we recall that the value of the bubble at time $t$ is given by $\hat{B}_{t} = N_{t}b_{1t} + N_{t-1}b_{2t}$. Let $B_{t} \equiv \hat{B}_{t}/k_{t}$. Using (10), (22)-(24), the evolution of the population size and the definition of $\lambda_{t}$, an intertemporal equilibrium is defined by:

$$B_{t} = n(\gamma_{t}, \lambda_{t})\hat{b}_{1}(\gamma_{t+1}, \lambda_{t+1}) + \hat{b}_{2}(\gamma_{t}, \lambda_{t}) \quad (26)$$

$$B_{t+1} = \frac{\alpha A}{n(\gamma_{t}, \lambda_{t})\gamma_{t+1}}B_{t} \quad (27)$$

$$\lambda_{t+1} = \frac{1}{\gamma_{t+1}}\lambda_{t} \quad (28)$$

This system drives the dynamics of $(\gamma_{t}, B_{t}, \lambda_{t}) \in \mathbb{R}^{3}_{++}$ for all $t$ and allows us to study the long run equilibrium.

Before, note that $B_{t}$, which represents the price of the speculative asset detrended by the capital-labor ratio times the population size, is determined by expectations on the future and there is a bubble if and only if $B_{t} > 0$. Therefore, this variable is not predetermined. The growth factor $\gamma_{t}$ is also a non-predetermined variable, because it depends on the capital-labor ratio $k_{t}(= \gamma_{t}k_{t-1}) = K_{t}/L_{t}$ and $L_{t} = N_{t}(1 - \psi) + N_{t-1}$ is not predetermined as it depends on the endogenous number of children $n_{t} = N_{t}/N_{t-1}$ chosen at period $t$. On the contrary, because $k_{t}$ implies that $\gamma_{t}$ is not predetermined, $\lambda_{t} = \epsilon/k_{t}$ is predetermined.

5 Asymptotic bubbly BGP

We focus on equilibria with endogenous growth. Along such a dynamic path, $\lambda_{t}$ decreases and tends to 0 when time tends to $+\infty$. Taking into account that $\lambda_{t} = \epsilon/k_{t}$, the dynamic system (26)-(28) admits no steady state, but may converge to a long run equilibrium with $\lambda^{*} > 0$, without attaining it. Such an equilibrium corresponds to an asymptotic BGP with a positive bubble and endogenous growth. Using equations (26)-(28), it is a stationary solution $(\gamma^{*}, B^{*}, \lambda^{*})$ satisfying $\lambda^{*} = 0$ and:

$$\gamma^{*}n(\gamma^{*}, 0) = \alpha A \quad (29)$$

$$B^{*} = n(\gamma^{*}, 0)\hat{b}_{1}(\gamma^{*}, 0) + \hat{b}_{2}(\gamma^{*}, 0) > 0 \quad (30)$$

Using equations (24) and (29), the growth factor $\gamma^{*}$ is given by:

$$\gamma^{*} = \frac{\alpha A}{\mu} \left(\psi(1 + \beta + \beta^{2} + \mu) - \mu\right) \quad (31)$$
The inequality $\psi > \mu / (1 + \beta + \beta^2 + \mu)$ ensures that $\gamma^* > 0$. Then, there is endogenous growth ($\gamma^* > 1$) if and only if:

$$A > \frac{\mu}{\alpha[\psi (1 + \beta + \beta^2 + \mu) - \mu]} \equiv A_1$$  \hspace{1cm} (32)

Using (22), (23), (29) and (31), the population growth factor is given by:

$$n(\gamma^*, 0) = \frac{\alpha A}{\gamma^*} = \frac{\mu}{\psi (1 + \beta + \beta^2 + \mu) - \mu} \equiv n^*$$  \hspace{1cm} (33)

and the shares of the bubble bought when young and adult by:

$$\tilde{b}_1(\gamma^*, 0) = (1 - \alpha) A - \frac{\psi}{\mu} A \left[ 1 + \alpha (\beta + \beta^2) \right] \equiv \tilde{b}_1^*$$  \hspace{1cm} (34)

$$\tilde{b}_2(\gamma^*, 0) = \frac{\psi \beta (1 - \alpha) A}{\psi (1 + \beta + \beta^2 + \mu) - \mu} \equiv \tilde{b}_2^*$$  \hspace{1cm} (35)

We deduce the value of the bubble over the capital-labor ratio times the population size:

$$B^* = n^* \tilde{b}_1^* + \tilde{b}_2^* = \frac{A \left[ 1 + \alpha \beta + (2\alpha - 1) \beta^2 \right]}{\psi (1 + \beta + \beta^2 + \mu) - \mu} \left( \psi_b - \psi \right)$$  \hspace{1cm} (36)

with

$$\psi_b \equiv \frac{\mu}{1 + \alpha \beta + (2\alpha - 1) \beta^2}$$  \hspace{1cm} (37)

Then, we can show the following:

**Proposition 2** Let

$$\psi \equiv \frac{\mu}{1 + \beta + \beta^2}$$  \hspace{1cm} (38)

Under $\alpha < \frac{\beta + 2\beta^2}{1 + 2\beta + 3\beta^2}$, there is a unique asymptotic BGP with endogenous growth ($\gamma^* > 1$, $n^* < 1/\psi$) and a positive bubble ($B^* > 0$) if $A > A_1$ and $\psi < \psi < \min\{\psi_b, 1\}$. In addition, one monotonically converges to this BGP.

**Proof.** See Appendix A. \hspace{1cm} $\blacksquare$

This proposition establishes the existence of a unique asymptotic BGP with a positive bubble and endogenous growth. Of course, there is endogenous growth under a sufficiently high productivity $A$, but it also requires a high enough $\psi$ (see equation (31)). Indeed, a high time cost per child $\psi$ reduces the incentive to have children. This implies that only a small amount of adult time is devoted to the total time cost of rearing children $\psi n^*$ (see equation (33)) and a large part of household resources can be used to invest in the productive asset, which promotes growth.

As we have seen previously, the shares of the bubble hold by both young and middle-age households, and the number of children decrease with the growth
factor. This implies that a positive bubble requires a not too significant growth. As a result, the cost $\psi$ should not be too high.

We now investigate more deeply the properties of this asymptotic bubbly BGP. We start by focusing on whether young and middle-age households are buyer ($\tilde{b}_1^* > 0$) or rather short-sellers ($\tilde{b}_1^* < 0$) of the speculative asset. By direct inspection of equation (35), there is no doubt that $\tilde{b}_2^* > 0$ at the bubbly BGP. Adult households use the bubble to transfer purchasing power to their last period of life. In contrast, whether young agents are buyers or short-sellers of the bubble needs a deeper analysis (see equation (34)).

Corollary 1 Let

$$\hat{\psi} \equiv \frac{(1 - \alpha)\mu}{1 + \alpha(\beta + \beta^2)}$$

Under $A > A_1$, the asymptotic BGP with endogenous growth and positive bubble is characterized by the following:

1. If \( \frac{\beta + \beta^2}{1 + 2\beta + 2\beta^2} \leq \alpha < \frac{3\beta + 3\beta^2}{1 + 2\beta + 2\beta^2} \), young agents are short-sellers ($\tilde{b}_1^* < 0$) for all $\hat{\psi} < \psi < \min\{\psi_b; 1\};$
2. If $\alpha < \frac{\beta + \beta^2}{1 + 2\beta + 2\beta^2}$, young agents are short-sellers ($\tilde{b}_1^* < 0$) for $\hat{\psi} < \psi < \min\{\psi_b; 1\}$, neither buy nor sell the bubble ($\tilde{b}_1^* = 0$) for $\psi = \hat{\psi}$, and buy the bubble ($\tilde{b}_1^* > 0$) for $\hat{\psi} < \psi < \hat{\psi}$. 

Proof. See Appendix B. ■

A direct implication of this result is that the existence of a bubbly BGP does not always require $\tilde{b}_1^* < 0$. An asymptotic bubbly BGP may exist if the young households buy the bubble and are not short-sellers of this asset to finance productive investment ($\tilde{b}_1^* > 0$).

Corollary 1 shows that young households are short-sellers of the bubble if either the return of the productive investment $\alpha A$ or the time cost per child $\psi$ is sufficiently significant. In the first case, the high return of capital creates an incentive to finance productive investment by selling short the speculative asset. In the second case, the quite large time cost per child incites households to have only a few number of children. Therefore, because of log-linear preferences, $\psi n^*$ is relatively low (see equation (33)). This low total time cost of rearing children incites the households to borrow when young to foster productive investment and redistributes income from the middle to the young age.

6 Is a bubble productive?

We analyze now whether a bubble is productive. We say that it is productive when the growth factor at the asymptotic bubbly BGP is higher than at the bubbleless one. In this case, the positive valuation of the bubble raises the growth rate.
For such an analysis, the conditions for the existence of a bubbleless BGP, i.e. $1 > \psi > \psi_{nwb}$ and $\mu < \mu_{nwb}$, and those for the existence of an asymptotic bubbly BGP, i.e. $\alpha < \frac{\beta + 2\beta^2}{\beta + 3\beta^2}$, $A > A_1$ and $\psi < \psi < \min\{\psi_b; 1\}$, should all be satisfied and should not imply an empty admissible interval for some parameters. Then, we could compare the asymptotic bubbly and bubbleless BGP. We show the following:

**Proposition 3** Let

$$\psi_p \equiv \frac{\mu[(1 - \alpha)(1 + \mu + \beta) - \alpha\beta^2]}{(1 + \alpha\beta)(1 + \beta + \beta^2 + \mu)}$$  \hspace{1cm} (40)

If $A > A_1$, $\mu < \mu_{nwb}$ and $\alpha \leq \frac{\beta}{1 + 2\beta + 3\beta^2}$, we have:

1. For $\max\{\psi; \psi_{nwb}\} < \psi \leq \psi_p$, the growth factor at the asymptotic bubbly BGP is lower than at the bubbleless BGP ($\gamma^* \leq \gamma_{wb}$);

2. For $\psi_p < \psi < \min\{\psi_b; 1\}$, the growth factor at the asymptotic bubbly BGP is strictly larger than at the bubbleless BGP ($\gamma^* > \gamma_{wb}$).

**Proof.** See Appendix C.  

As it is illustrated on Figure 1, Proposition 3 means that when the time cost per child $\psi$ is relatively low, the economy without bubble is characterized by a higher growth rate than the economy with a bubble. On the contrary, when the time cost per child $\psi$ is relatively high, the economy with bubble is characterized by a higher growth rate than the economy without bubble. In other words, if the time cost per child $\psi$ is lower than the threshold $\psi_p$, the bubble is damaging for growth. If it is larger, the bubble or the positive valuation of the speculative asset is beneficial for growth. This last result is in accordance with the empirical facts underlying that episodes of bubble are associated with larger growth rates, as it is well documented in Caballero et al. (2006) or Martin and Ventura (2012).
While seminal papers like Tirole (1985) show the bubbles imply lower capital accumulation, various more recent contributions provide some mechanisms that reconcile the existence of rational bubbles with larger levels of capital (Fahri and Tirole (2012), Kocherlakota (2009), Martin and Ventura (2012), Miao and Wang (2011), Raurich and Seegmuller (2015)). In models with endogenous growth, Grossman and Yangawa (1993) show that the existence of a bubble reduces the growth rate, but more recent results show that bubbles may be in accordance with higher growth rates. Olivier (2000) highlights that a bubble on equity raises the value of firms, which promotes firm creation and growth. On the contrary, bubbles on unproductive assets have the same effect than in Grossman and Yangawa (1993). Hirano and Yangawa (2013) discuss the existence of bubbles in a model with heterogeneous investment projects and borrowing constraints. These authors are especially concerned with the interplay between growth and the existence of bubbles according to the degree of financial imperfections.

Our paper differs from these last two contributions. In contrast to Olivier (2000), growth is enhancing in our framework even if there is a bubble on an unproductive asset. The mechanism that allows to have a growth enhancing or productive bubble in our framework is different. It is based on the demographic aspects and the distribution of income at the different ages (young, adult, old). We also depart from Hirano and Yangawa (2013) since we do not consider heterogeneous investment projects and do not discuss the results according to the level of financial frictions.

It is interesting to note that we can have \( \psi_p \) lower than \( \hat{\psi} \).\(^6\) By inspection of Corollary 1 and Proposition 3, this means that for \( \psi_p < \psi < \hat{\psi} \), we can have a higher growth of the capital-labor ratio at the asymptotic bubbly BGP than at the bubbleless one \( (\gamma^* > \gamma_{wb}) \) even if young households are not short-sellers of the speculative asset. This is in contrast with Raurich and Seegmuller (2015) who study the existence of rational bubbles in an overlapping generations economy with three period-lived households and exogenous growth. They show that capital is higher at the bubbly than at the bubbleless steady state only if young households are short-sellers of the bubble. Here, we obtain a different result, which means that the mechanism for the existence of a productive bubble is different than in Raurich and Seegmuller (2015).

Our model with endogenous fertility and rational bubble also allows us to refer to the debate on the link between population size and the value of asset (Abel (2001), Geanakoplos et al. (2004), Poterba (2005)), which explains that asset price is higher when the relative number of savers is more significant. In particular, Geanakoplos et al. (2004) consider an overlapping generations model with three-period lived agents and identify that the number of savers is relatively more important as the ratio of middle-age over young households is higher.

We contribute to this debate focusing on the price of the speculative asset. Obviously, the price of this asset is larger with the bubble than without. In

\(^6\)We note that when Proposition 3 applies, it corresponds to case 2 of Corollary 1, because

\[
\frac{\beta}{\tau + 2\beta + \beta^2} < \frac{\beta + \beta^2}{\tau + 2\beta + 2\beta^2}.
\]

Then, using (39) and (40), \( \psi_p < \psi \) requires \( \mu < \frac{1 - \alpha (1 - \alpha) + \alpha^2 \beta (\beta + \beta^2)}{\alpha (1 - \alpha)} \).
the next corollary, we first make sure that, at the asymptotic bubbly BGP, middle-age households hold a larger amount of the bubble than young ones. Then, observing that the ratio of middle-age over young households is given by

\[ \frac{N_{t-1}}{N_t} = \frac{1}{n_t} \]

we note \( \frac{N_{t-1}}{N_t} = \frac{1}{n_t} \), we note \( MY_t \equiv \frac{N_{t-1}}{N_t} = \frac{1}{n_t} \), we note \( MY_t^* \) and \( MY_{wb}^* \) the value of this ratio evaluated at the asymptotic bubbly and bubbleless BGPs, respectively, and are able to compare them:

**Corollary 2** If \( A > A_1, \mu < \mu_{wb}, \psi_p < \psi < \min\{\psi_0; 1\} \) and \( \alpha \leq \frac{\beta}{1+2\beta+\beta^2} \), we have \( \hat{b}_2^* > n^*\hat{b}_1^* \); \( \gamma^* > \gamma_{wb} \) and \( n^* < n_{wb} \), which means that \( MY^* > MY_{wb}^* \).

**Proof.** See Appendix D. \( \blacksquare \)

This corollary ensures first that middle-age households hold a larger amount of the bubble than young agents living at the same period (\( \hat{b}_2^* > n^*\hat{b}_1^* \)). Therefore, a higher \( MY \) is a relevant measure to argue that more traders buy the speculative asset. Then, comparing the asymptotic bubbly and bubbleless BGPs, we have not only that \( \gamma^* > \gamma_{wb} \), but also \( MY^* > MY_{wb}^* \) and the price of the speculative asset is of course larger at the asymptotic bubbly BGP, since it is zero at the bubbleless one. Hence, following a financial crisis, which means that agents coordinate their expectations on the bubbleless BGP,\(^7\) we may not only get a lower growth factor, as already discussed in Proposition 3, but the decrease of the price of the speculative asset (the crash of the bubble) occurs at the same time than a decrease in the ratio of middle-age over young households. This corroborates the theoretical and empirical findings of Geanakoplos et al. (2004) on the link between demography and asset prices. The basic mechanism they focus on is the following: the lower price of asset is explained by a relative lower number of savers, measured by a smaller ratio of adult over young households, which reduces the demand of asset. In our model, we associate the asset price to the bubble on the speculative asset and a larger ratio of adult over young households means that the main buyers of the speculative asset are relatively more. Even if we observe the same link than Geanakoplos et al. (2004) between the asset price and the demography, it should be interpreted in a different way: it is the market crash of the bubble that explains the smaller ratio of middle-age over young rather than the opposite.

### 7 Crowding-out versus crowding-in effect

To understand the economic mechanism that allows to have larger growth when there is a bubble, we highlight three main ingredients that will help us to understand what happens according to the value of the time cost per child \( \psi \).

1. The speculative asset allows to smooth consumption along the life-cycle. At middle-age, an household buys the bubble (\( \hat{b}_2^* > 0 \)) to transfer purchasing power to the old age. Using \( \hat{b}_1^* \), the household smooths consumptions and incomes between young and adult ages. As already discussed in

\(^7\)A financial crisis can be associated to a market crash of the bubble. One can refer to Weil (1987) who introduces a probability of bubble crash according to a sunspot process.
the previous section, we learn from Corollary 1 and Proposition 3 that a growth enhancing bubble at the BGP is not equivalent to have $\tilde{b}_1^* < 0$, i.e. young households are short-sellers of the bubble. Indeed, there is a value $\hat{\psi}$ such that $\tilde{b}_1^* > 0$ if $\psi < \hat{\psi}$, $\tilde{b}_1^* = 0$ if $\psi = \hat{\psi}$, while $\tilde{b}_1^* < 0$ if $\psi > \hat{\psi}$.

2. In Corollary 2, we have shown that if $\gamma^* > \gamma_{wb}$, we get $n^* < n_{wb}$. Now, without comparing the growth factors, we deduce, using (16) and (33), that $n^* < n_{wb}$ if and only if $\psi > \psi_n$, with:

$$\psi_n \equiv \mu \left(1 - \alpha \right) \frac{1 + \beta}{1 + \alpha \beta}.$$ (41)

3. Recall now that $\gamma$ denotes the growth factor of the capital-labor ratio. It is often more usual to define the growth factor of capital per capita (or GDP per capita). This last one is defined by $K_t/N_t = k_t L_t/N_t = k_t(1 - \psi + 1/n_t)$. On a BGP, since $n$ is constant, the growth factor of capital per capita is also equal to $\gamma$ and the growth factor of capital is defined by:

$$\gamma_K \equiv \frac{K_{t+1}}{K_t} = \frac{k_{t+1}}{k_t} \frac{N_{t+1}}{N_t} = \gamma n$$

Using (29), $\gamma_K = \alpha A \equiv \gamma_K^*$ at the asymptotic bubbly BGP and, using (14) and (16), $\gamma_K = \frac{K_{t+1}^{\alpha A} N_{t+1}^{\gamma A}}{k_{t+1}^{\alpha A} N_t^{\gamma A}} \equiv \gamma_{K, wb}$ at the bubbleless BGP. We deduce that $\gamma^*_K > \gamma_{K, wb}$ if and only if $\psi > \psi_K$, with:

$$\psi_K \equiv \frac{\mu (1 - \alpha)}{1 + \alpha \beta}.$$ (42)

Using (39), (41) and (42), we have that $\psi_n < \tilde{\psi} < \psi_K$. This allows us to draw Figure 1, which is useful to understand why the bubble raises or not.
growth of capital per capita. There are three main configurations according to the value of $\psi$: $\psi < \psi_n$, $\psi_n < \psi < \psi_K$ and $\psi > \psi_K$.

Before studying these different configurations, it is useful to note that since $\gamma = \gamma_K/n$, the threshold value $\psi_p$ above which $\gamma^* > \gamma_w$ belongs to $(\psi_n, \psi_K)$. Of course, $\gamma^* > \gamma_w$ for $\psi \geq \psi_K$ and $\gamma^* < \gamma_w$ for $\psi \leq \psi_n$.

1. $\psi < \psi_n$: Crowding-out effect
   Since $\psi$ is quite low, we have seen that the total time cost of rearing children $\psi_n$ is relatively large, whether or not the bubble exists. The labor income at middle-age is relatively low, because of a weak labor supply. When the speculative asset is positively valued, young households buy the bubble to transfer income to the middle-age. Therefore, they invest less in capital when there is a bubble. Growth is lower when the bubble exists. This also implies that the cost of having children $\psi w_{t+1}$ in terms of the consumption good grows slowly, which explains that population growth is larger at the asymptotic bubbly BGP.

2. $\psi_n < \psi < \psi_K$: Indeterminate crowding effect
   Both $\psi$ and $\psi_n^*$ take intermediate values. The main mechanism at stake is clearly different than in the previous configuration. To understand what happens, let us assume $\psi = \hat{\psi}$. At the asymptotic bubbly BGP, young households neither buy, nor sell the bubble, i.e. $b^*_1 = 0$. Since $b^*_2 > 0$ to finance consumption when old, a middle-age household needs to have less expenditures for children, implying a lower population growth at the asymptotic bubbly BGP. Then, productive investment $a$ can be larger or lower at the asymptotic bubbly BGP than at the bubbleless one because of two opposite effects: On the one hand, fewer children expenses have to be covered; On the other hand, the purchase of the bubble to finance consumption when old has to be financed.

3. $\psi > \psi_K$: Crowding-in effect
   As we have seen, since $\psi$ is large enough, $\psi_n$ is relatively low, whether or not the bubble exists. Therefore, the main mechanism is exactly the opposite one than in the first configuration. The labor income when adult is relatively significant. When the bubble exists, there is a transfer of resources from the adult to the young age. The young household becomes a short-seller of the bubble and increases productive investment. Therefore, growth is larger at the asymptotic bubbly BGP. This also induces a larger cost of having children $\psi w_{t+1}$ in terms of the consumption good. Therefore, population growth is lower when there is a bubble.

8 Concluding remarks

We develop a model where a speculative asset (bubble) is used to finance two types of expenditures, productive investment and children. If the time cost per child $\psi$ is low, the household has a relatively significant number of children,
meaning that the total time cost $\psi n$ is high. In this case, the bubble is mainly used to finance children instead of productive capital. Growth is lower. If $\psi$ is high, we have the opposite. The total time cost of having children $\psi n$ is lower. The bubble is mainly used to finance productive investment, enhancing growth.

The comparison between equilibria with and without bubble can also be interpreted in terms of financial development. Such an interpretation of our results may be interesting regarding the literature that has studied the link between financial development and growth.\(^8\) In our framework, the bubbleless equilibrium can be seen as the outcome of an economy where the only asset is capital, whereas the bubbly one as resulting from an economy where financial markets are more developed since there is an adding traded asset valued on such a market. Accordingly, we can say that the bubbleless economy is characterized by a less significant financial development than the bubbly economy. Therefore, our model shows that financial development is beneficial for growth only if $\psi$ is high enough. On the contrary, when $\psi$ is sufficiently low, financial development is damaging for growth. Hence, on the one hand, ours conclusions mitigate the enhancing role of financial development on economic growth. On the other hand, we establish a new mechanism that contributes to explain the relationship between financial development and growth.

Finally, we can ask whether a country is characterized by a quite high time cost $\psi$. We note that in our framework, education is not endogenized, but $\psi$ can be seen as a measure of a time cost for education. Even if we have assumed that it is constant, data shows that it strongly varies across countries. It is even argued that this cost is higher in countries with higher income. For instance, Guryan et al. (2008) find a positive correlation between average time spent on child care (after purging for demographic differences across the countries) and GDP per capita. This allows us to associate a high $\psi$ to more developed countries and a low $\psi$ to less developed ones. Then, we can deduce from our previous discussion that, for a country sufficiently developed ($\psi > \psi_p$), financial development is growth enhancing, while for a less developed country ($\psi < \psi_p$), financial development is damaging for growth.

Appendix

A Proof of Proposition 2

We have $n^* < 1/\psi$ if and only if $\psi > \bar{\psi}$, where this inequality ensures that $\psi > \mu/(1 + \beta + \beta^2 + \mu)$. Then, using (36), we immediately see that $B^* > 0$ if and only if $\psi < \psi_b$. We also note that $\psi < 1$ because $\mu < 1$ and $\psi < \psi_b$ is equivalent to $\alpha < \frac{1}{1+2\beta+3\beta^2}$. Of course, there is endogenous growth because $A > A_1$.

Let us focus now on the stability properties. Even if $(\gamma^*, B^*, 0)$ is an asymptotic BGP, it is important to note that from the mathematical point of view,

\(^8\)See Levine (2005) for a survey and Madsen and Ang (2016) for a recent contribution.
\((\gamma_t, B_t, \lambda_t) = (\gamma^*, B^*, 0)\) is a steady state of the dynamic system (26)-(28). Therefore, to analyze the stability properties of the equilibrium \((\gamma^*, B^*, 0)\), we use standard mathematical tools.

Differentiating the dynamic system (26)-(28) in the neighborhood of the BGP \((\gamma^*, B^*, 0)\), we obtain a linear system of the following form:

\[
\begin{align*}
\frac{d}{dt}B_{t+1} &= a_{11}d_{B_t} + a_{12}d_{\gamma_t} + a_{13}d_{\lambda_t} \\
\frac{d}{dt}\gamma_{t+1} &= a_{21}d_{B_t} + a_{22}d_{\gamma_t} + a_{23}d_{\lambda_t} \\
\frac{d}{dt}\lambda_{t+1} &= a_{33}d_{\lambda_t}
\end{align*}
\]

Therefore, the characteristic polynomial can be written\( P(\eta) \equiv (a_{33} - \eta)(\eta^2 - T\eta + D) = 0\), with \(T = a_{11} + a_{22}\) and \(D = a_{11}a_{22} - a_{12}a_{21}\). We immediately deduce that one eigenvalue is given by \(\eta_1 = a_{33} = 1/\gamma^* \in (0, 1)\). To determine the other eigenvalues, we compute the terms \(a_{ij}\), with \(\{i, j\} = \{1, 2\}\):

\[
\begin{align*}
a_{11} &= 1 - \frac{B^*}{\alpha A \partial \tilde{b}_1^*/\partial \gamma_{t+1}} \\
a_{12} &= \frac{B^* n^*}{n^* \partial \gamma_t} + \frac{B^*}{\alpha A \partial \tilde{b}_1^*/\partial \gamma_{t+1}} \left( \tilde{b}_1^* \partial n_t/\partial \gamma_t + \tilde{b}_2^* \partial n_t/\partial \gamma_t \right) \\
a_{21} &= \frac{\gamma^*}{\alpha A \partial \tilde{b}_1^*/\partial \gamma_{t+1}} \\
a_{22} &= -\frac{\gamma^*}{\alpha A \partial \tilde{b}_1^*/\partial \gamma_{t+1}} \left( \tilde{b}_1^* \partial n_t/\partial \gamma_t + \tilde{b}_2^* \partial n_t/\partial \gamma_t \right)
\end{align*}
\]

The minor \(D\) is given by:

\[
D = \frac{\gamma^*}{\alpha A \partial \tilde{b}_1^*/\partial \gamma_{t+1}} \left( \frac{B^* n^*}{n^* \partial \gamma_t} - \tilde{b}_1^* \partial n_t/\partial \gamma_t - \tilde{b}_2^* \partial n_t/\partial \gamma_t \right)
\]

Using \(n^* = \alpha A/\gamma^*\) and \(B^* = n^* \tilde{b}_1^* + \tilde{b}_2^*\), we obtain:

\[
D = \frac{1}{(n^*)^2 \partial \tilde{b}_1^*/\partial \gamma_{t+1}} \left( \tilde{b}_2^* \partial n_t/\partial \gamma_t - \tilde{b}_2^* n^* \right)
\]

From (23) and (24), we get:

\[
\begin{align*}
\frac{\partial n_t}{\partial \gamma_t} &= -\frac{\mu/\psi}{1 + \beta + \beta^2 + \mu} \frac{\alpha A}{\gamma_t^2} \\
\frac{\partial \tilde{b}_2^*}{\partial \gamma_t} &= -\frac{\beta^2 \alpha A}{1 + \beta + \beta^2 + \mu} \frac{(1 - \alpha) A}{\gamma_t^2}
\end{align*}
\]

Using (33) and (35), we deduce that \(D = 0\). This means that one eigenvalue is zero, i.e. \(\eta_2 = 0\). Using now (23), (24), (A.1) and (A.4), \(T\) rewrites:

\[
T = 1 - \frac{B^*}{\alpha A (\gamma^*/\gamma^* + 1)} \partial \tilde{b}_1^*/\partial \gamma_{t+1}
\]

16
We easily derive from (22) that:
\[
\frac{\partial \hat{b}^*_1}{\partial \gamma_{t+1}} = -1 - \frac{(1 - \psi)\mu/\psi}{1 + \beta + \beta^2 + \mu} - \frac{1}{1 + \beta + \beta^2 + \mu} \frac{1 - \alpha}{\alpha} < 0
\]
We deduce that the last eigenvalue is given by \( \eta_3 = T > 1 \).

B Proof of Corollary 1

Using (34), we see that \( \hat{b}^*_1 < 0 \) if and only if \( \hat{\psi} > \hat{\psi} \). We further have that \( \hat{\psi} < \psi_b, \hat{\psi} < 1 \) is ensured by \( \mu < 1 \) and \( \hat{\psi} > \psi \) if and only if \( \alpha < \frac{\beta + \beta^2}{1 + 2\beta + 3\beta^2} \). If this last inequality is not satisfied, we have \( \hat{\psi} \leq \psi \). Since \( \frac{\beta + \beta^2}{1 + 2\beta + 3\beta^2} < \frac{\beta + 2\beta^2}{1 + 2\beta + 3\beta^2} \), the corollary immediately follows.

C Proof of Proposition 3

We first take into account the conditions for the existence of a bubbleless BGP, i.e. \( \mu < \mu_{nwb} \) and \( \psi > \psi_{nwb} \). They are compatible with the conditions for the existence of an asymptotic bubbly BGP if \( \psi_{nwb} < \psi_b \), which is ensured if and only if \( \alpha \leq \frac{\beta}{1 + 2\beta + 3\beta^2} \). Therefore, the bubbleless and asymptotic bubbly BGP coexist if \( \max\{\psi; \psi_{nwb}\} < \psi < \min\{\psi_b; 1\} \).

Using (14) and (31), we note that \( \gamma^* > \gamma_{wb} \) if and only if \( \psi > \psi_p \). We can further show that \( \psi_p < 1 \) and \( \psi_p < \psi_b \) because \( \alpha \leq \frac{\beta}{1 + 2\beta + 3\beta^2} \). Moreover, Using (17), (38) and (40), we can further show that under \( \alpha \leq \frac{\beta}{1 + 2\beta + 3\beta^2} \), we have \( \psi_p > \max\{\psi; \psi_{nwb}\} \). Indeed, \( \psi_p > \psi_s \) is equivalent to:
\[
(1 + \beta + \beta^2)[\beta - \alpha(1 + 2\beta + \beta^2)] > \mu(\alpha(1 + 2\beta + \beta^2) - (\beta + \beta^2)]
\]
and \( \psi_p > \psi_{nwb} \) to:
\[
\mu[\beta - \alpha(1 + 2\beta)] > (1 + \beta)[\alpha(1 + 2\beta + \beta^2) - \beta]
\]
Both these inequalities are satisfied under \( \alpha \leq \frac{\beta}{1 + 2\beta + 3\beta^2} \). This means that \( \max\{\psi; \psi_{nwb}\} < \psi_p < \min\{\psi_b; 1\} \) and the proposition follows.

D Proof of Corollary 2

Using (33)-(35), the inequality \( \hat{b}^*_2 > n^* \hat{b}^*_1 \) is equivalent to \( \psi > \psi_s \), with:
\[
\psi_s \equiv \frac{\mu(1 - \alpha)}{1 + \alpha\beta + \beta^2}
\]
Using now (40), \( \psi_p > \psi_s \) is equivalent to:
\[
\beta - \alpha(1 + 2\beta + \beta^2) + \mu(1 - \alpha) > 0
\]
which is always satisfied for \( \alpha \leq \frac{\beta}{1 + 2\beta + 3\beta^2} \).

Using Proposition 3, we have \( \gamma^* > \gamma_{wb} \). Using (15) and (24) with \( \lambda_{t+1} = 0 \), we easily deduce that \( n^* < n_{wb} \). This means that \( MY^* > MY_{wb} \).
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