Manipulability and tie-breaking in constrained school choice

Benoit Decerf
CORE, Université Catholique de Louvain
benoit.decerf(at)uclouvain.be

Martin Van der Linden
Department of Economics, Vanderbilt University
martin.van.der.linden(at)vanderbilt.edu

August 5, 2016

Abstract

In constrained school choice mechanisms, students can only rank a subset of the schools they could potentially access. We characterize dominant and undominated strategies in the constrained Boston (BOS) and deferred acceptance (DA) mechanisms. Using our characterization of dominant strategies we show that in constrained DA, the single tie-breaking rule outperforms the multiple tie-breaking rule in terms of both manipulability and stability. We also show that DA is less manipulable than constrained BOS in the sense of Arribillaga and Massó (2015). Using our characterizations of undominated strategies, we derive advice for the students and show that more strategies can be excluded on the basis of dominance in constrained DA than in constrained BOS.

JEL Classification: C78, D47, D82, I 20.
Keywords: School choice, Dominant strategy, Undominated strategy, Manipulability, Stability, Tie-breaking Boston mechanism, Deferred acceptance mechanism.

1 Introduction

In the problem of assigning students to schools, the deferred-acceptance mechanism (DA) is non-manipulable and stable whereas the Boston mechanism (BOS) satisfies none of the two properties (Abdulkadiroğlu and Sönmez, 2003). These classical results are often used to argue in favor of DA and against BOS.

One limitation of these results is they require that students be able to report a ranking of all the schools they could potentially access. In practice however, most school districts use constrained mechanisms (Haerengen and Klijn, 2009) in which students are only allowed to rank a limited number of schools.\footnote{Stability requires that all students prefer their assignment to any school at which they have a higher priority than a student assigned to the school.}

\footnote{For example, at the time Haerengen and Klijn (2009) was written, the authors reported that the New York City school district allowed students to rank only 12 programs, while the district had more than 500 different programs available.}

1 Stability requires that all students prefer their assignment to any school at which they have a higher priority than a student assigned to the school.

2 For example, at the time Haerengen and Klijn (2009) was written, the authors reported that the New York City school district allowed students to rank only 12 programs, while the district had more than 500 different programs available.
denote the constrained version of the two above mechanisms by $DA^{k}$ and $BOS^{k}$, where $k$ is the number of schools students can report.

Unfortunately, when it is constrained, $DA$ looses both of its appealing properties. First, $DA^{k}$ is manipulable because students have to worry about running out of reported schools if they rank schools at which they have a low priority. Second, $DA^{k}$ is unstable when students cannot rank all the schools because they may fail to claim a seat at some schools that they like better than their assignment. Hence, the comparison of $DA^{k}$ and $BOS^{k}$ is much less clear than that of $DA$ and $BOS$.

In this paper, we characterize the undominated and dominant strategies of the games induced by $DA^{k}$ and $BOS^{k}$. We use these characterizations to show that the comparison between $DA$ and $BOS$ extends to $DA^{k}$ and $BOS^{k}$ in the sense that $DA^{k}$ has better stability and manipulability properties than $BOS^{k}$.

First, we demonstrate that the proportion of students who have a dominant strategy in $DA^{k}$ increases with $k$. This result is not the mere consequence of more students being able to rank all their acceptable schools as $k$ increases. Instead, it relates to possible correlations between priorities at schools and to the concept of a safe set of school that we introduce. We also show that students who have a dominant strategy in $DA^{k}$ do not cause instabilities. These two results suggest that stability and non-manipulability improve in $DA^{k}$ as $k$ increases.

The same is not true for $BOS^{k}$ where dominant strategies are not affected by $k$ or by correlation in the priorities. In fact, given any profile of priorities, every preference relation that provides a student with a truthful dominant strategy in $BOS^{k}$ also provides the student with a truthful dominant strategy in $DA^{k}$ (but the converse is not true). Thus $DA^{k}$ is less manipulable than $BOS^{k}$ in the sense of Arribillaga and Massó (2015). Using the same criterion, we show that contrary to $BOS^{k}$ which is equally manipulable for all $k$, $DA^{k}$ becomes less manipulable as $k$ increases.

Using our characterization of undominated strategies, we derive recommendations on the way student should report their preferences in $DA^{k}$ and $BOS^{k}$. Because these recommendations are based on dominance only, they are uncontroversial in the sense that they do not depend on the preferences reported by the other students.

Roughly, we show that in $DA^{k}$ students should report as many acceptable schools as possible without switches (i.e., in the same order as their true preferences). But students should also pay careful attention to the priority structure. Indeed, not all strategies that report as many acceptable schools as possible without switch are undominated. Students must take advantage of safe sets of schools and select the right combination of schools to report for their strategy to be undominated.

In $BOS^{k}$, it is much harder to give uncontroversial recommendations to students. In fact, we show that the set of strategies that can be ruled out on the basis of dominance in $BOS^{k}$ is a subset of the set of strategies that can be ruled out on the same basis in $DA^{k}$. In this sense too, $BOS^{k}$ is strategically more involved than $DA^{k}$.

Another limitation of the standard theory of school choice is the assumption that schools rank students according to a strict priority order. In practice, the

---

3 I.e., students who play a dominant strategy prefer their assignment (i) to being unassigned, (ii) to any school at which they have a higher priority than a student assigned to the school, and (iii) to any school that still has available seats.
criteria used to establish priorities are often too sparse to establish such a strict ranking. To apply standard assignment mechanisms, school district therefore rely on random tie-breaking rules.

The two most common tie-breaking rules are the single tie-breaking rule (STB) and the multiple tie-breaking rule (MTB). STB breaks ties in the same way in every school whereas MTB draws a different tie-breaking order at each school.

Counter-factual simulations based on field data as well as theoretical results in large random environments suggest that DA is more efficient when used with STB than with MTB. In contrast with the existing literature, we introduce the first results on the stability and manipulability effects of tie-breaking rules. Intuitively, STB induces more correlation than MTB among priorities at different schools (on average). Thus, by the results described above, STB makes it more likely for students (i) to have a dominant strategy and (ii) not to cause instabilities. We confirm this intuition in a simulation using random profiles. Our simulation shows a clear increase in the proportion of students who have a dominant strategy as \( k \) increase when STB is used. The same proportion is almost flat when MTB is used.

2 The school choice model

The model is similar to Haeringer and Klijn (2009). There is a finite set of schools \( S := \{s_1, \ldots, s_m\} \) with \( m \geq 2 \), and a finite set of students \( T := \{t_1, \ldots, t_n\} \).

A generic school is denoted \( s_j \) or sometimes \( s \). Every school \( s_j \in S \) has a capacity \( q_j \) and a priority profile \( F_j \). A capacity \( q_j \in \mathbb{N}_+ \) represents the number of seats available at school \( s_j \). Priorities \( F_j \) are linear orderings on the students in \( T \). We assume that the seats are in short-supply, that is \( \sum_{s_j \in S} q_j < n \).

A generic student is denoted \( t_i \), or sometimes \( t \). Every student \( t_i \) has preferences \( R_i \). Preferences are a linear ordering on \( S \cup \{t_i\} \). A preference profile \( R := (R_1, \ldots, R_n) \) is a list of the preferences of every \( t_i \in T \). For a given preference profile \( R \), the list containing the preferences of everyone but \( t_i \) is \( R_{-i} \).

Profiles of preferences, priorities, and quotas are fixed but arbitrary throughout most of the paper. Exceptions are the simulation in Section 5.2 and the

---

4 In Boston for example, Abdulkadiroğlu et al. (2006) report that the high-school district can have more than 6000 applicants for only 5 priority groups. Thus, numerous ties ensue and tie-breaking plays an essential role in students’ eventual assignment.

5 See Abdulkadiroğlu et al. (2009) and De Haan et al. (2015) for simulations based on field data, and Ashlagi et al. (2015), Ashlagi and Nikzad (2015) and Arnosti (2015) for theoretical results in the large.

6 The model in this paper differs slightly from the model of the companion paper Decerf and Van der Linden (2016). Among other differences, here we take advantage of the fact that priorities, quotas and preferences can be fixed throughout most of the analysis to simplify the model.

7 This is a minor restriction which is common in the literature (see Decerf and Van der Linden (2016) for a detailed justification). We adopt the assumption throughout for simplicity. Most of our results on DA hold without the short-supply assumption. It is very useful in BOS however because it significantly restricts the set of undominated strategies (see Section 6). Even then, it is often enough to assume that no set of \( k \) schools has more seats than the number of students.
the cardinality of $A$ if $M$ is $Q$ $Q$ $S$ are ranked in $(iv)$

$\mu$ preferences $Q$ choice (Ergin and Sönmez, 2006). Therefore, we sometimes refer to reported

t (Non wastefulness) no student prefers a school with an available seat to her

and (Individual rationality) no student $\mu$ contains no blocking pairs,

(Non wastefulness) no student prefers a school with an available seat to her

assignment, that is there exists no $t_i \in T$ and $s_j \in S$ such that $s_j P_i \mu(t_i)$ and there are less than $q_j$ students assigned to $s_j$ in $\mu$.

A student $t_i$ causes an instability in assignment $\mu$ if she is involved in a

violation of one of the three above properties, i.e., $t_i$ is either (i) involved in a

blocking pair, (ii) assigned to an unacceptable school, or (iii) prefers a school

with an available seat to her assignment.

School choice mechanisms and games of school choice

A (school choice) mechanism $M$ associates every profile of reported preferences $Q := (Q_1, \ldots, Q_n)$ in some domain with a feasible assignment $\mu$. In a

constrained mechanism $M^k$, the domain only contains reported preference profiles in which students report no more than $k \leq m$ acceptable schools. The

notation and terminology for preferences extend to reported preferences : (i) $s Q_i s'$ means that $t_i$ reports $s$ weakly before $s'$ in $Q_i$ (where possibly $s = s'$), (ii) school $s \in S$ is ranked by $t_i$ in $Q_i$ if $s Q_i t_i$ (iii) $s \in Q_i$ if $s$ is ranked in $Q_i$, (iv) $\#Q_i$ is the number of ranked schools in $Q_i$, (v) $S \subseteq Q_i$ if all the schools in $S$ are ranked in $Q_i$, (vi) for all $x$, $Q_i(x)$ is the school ranked in $x$-th position in $Q_i$, (vii) a typical profile of reported preferences is $Q := (Q_1, \ldots, Q_n)$ and (viii) $Q_{-i}$ is the list of reported preferences of every student but $t_i$.

For any $Q$ and any $t_i$, the school $t_i$ is assigned to under assignment $M(Q)$ is $M_i(Q)$. Student $t_i$ is assigned in $M$ given $Q$ if $M_i(Q) \neq t_i$ and unassigned

if $M_i(Q) = t_i$.

A pair $(M, R)$ defines a strategic form game known as a game of school choice (Ergin and Sönmez, 2006). Therefore, we sometimes refer to reported preferences $Q_i$ as strategies. Given mechanism $M$, $Q_i$ is a dominant strategy if

$M_i(Q_i, Q_{-i}) R_i M_i(Q_i', Q_{-i})$, for all $Q_{-i}$ and all $Q_i'$.
Strategy $Q_i$ is an undominated strategy if for all $Q'_i$,

$$M_i(Q_i, Q_{-i}) = M_i(Q'_i, Q_{-i}), \quad \text{for all } Q_{-i}, \text{ or}$$

$$M_i(Q_i, Q_{-i}) \not\leq M_i(Q'_i, Q_{-i}), \quad \text{for some } Q_{-i}.$$ 

Strategy $Q_i$ is dominated if it is not undominated, that is there exists a strategy $Q'_i$ such that $M_i(Q'_i, Q_{-i}) \not\leq M_i(Q_i, Q_{-i})$ for all $Q_{-i}$ and $M_i(Q'_i, Q_{-i}) \not\geq M_i(Q_i, Q_{-i})$ for some $Q_{-i}$.

### 3 Two classes of competing mechanisms

In this section we describe the two classes of school choice mechanisms that we focus on. These classes were identified by Haeringer and Klijn (2009) and correspond to constrained versions of $BOS$ and $DA$. We first describe the well known unconstrained $BOS$.

**Round 1:** Students apply to the school they reported as their most preferred acceptable school (if any). Every school that receives more applications than its capacity starts rejecting the lowest applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are definitively accepted at the schools they applied to and capacities are adjusted accordingly.

$\vdots$

**Round $\ell$:** Students who are not yet assigned apply to the school they reported as their $\ell$th acceptable school (if any). Every school that receives more new applications in round $\ell$ than its remaining capacity starts rejecting the lowest new applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are definitively accepted at the schools they applied to and capacities are adjusted accordingly.

The algorithm terminates when all acceptable schools of the reported profile have been considered, or when every student is assigned to a school. The constrained versions of $BOS$ which we will denote by $BOS^k$ are identical to $BOS$ except that no student is allowed more than $k$ acceptable schools in her reported preferences.

We now turn to $DA$. Again, we first describe the famous unconstrained $DA$.

**Round 1:** Students apply to the school they reported as their most preferred acceptable school (if any). Every school that receives more applications than its capacity definitively rejects the lowest applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are temporarily accepted at the schools they applied to (this means they could still be rejected in a later round).

$\vdots$

**Round $\ell$:** Students who were rejected in round $\ell-1$ apply to their next acceptable school (if any). Every school considers the new applicants of
round ℓ together with the students it temporarily accepted. If needed, each school definitely rejects the lowest students in its priority ranking, up to the point where it meets its capacity. All other applicants are temporarily accepted at the schools they applied to (this means they could still be rejected in a later round).

The algorithm terminates when all acceptable schools of the reported profile have been considered, or when every student is assigned to a school. The constrained versions of DA which we will denote DA$k$ are identical to DA except that no student is allowed more than $k$ acceptable schools in her reported preferences.

4 Safe sets, safe strategies and other concepts

In this section, we introduce some concepts that will prove useful in the characterization of undominated and dominant strategies in DA$k$ and BOS$k$. Of particular importance are the concepts of safe sets and safe strategies which we illustrate in more details.

In many games, the set of undominated strategies is quite large. In some games, it is even equal to the whole strategy space. In school choice mechanisms however, requiring that a strategy be undominated is far from vacuous. In DA$k$ mechanisms in particular, eliminating dominated strategies constrains the set of strategies significantly. Lemma 1 in Section 5.1 shows that most undominated strategies

(i) do not rank unacceptable schools,

(ii) feature no switches, where a switch is a situation in which schools $s$ and $s'$ are ranked in $Q_i$ and $s Q_i s'$ although $s' P_i s$, and

(iii) rank as many acceptable schools as possible, i.e., $\#Q_i = \min\{k, \#R_i\}$.

When a strategy satisfies (i), (ii) and (iii) we say that this strategy is clean.

As we show in Example 1 below, although most undominated strategies are clean, not all clean strategies are undominated. The example relies on the concept of a safe set (of schools). A set of schools $S^* \subseteq S$ is safe for $t_i$ if whenever $t_i$ ranks all the schools in $S^*$, $t_i$ is at least assigned to the worst school in $S^*$ according to $t_i$’s reported preferences. Formally, take any set of schools $S^* \subseteq S$. Let $Q_i^*$ be any strategy that ranks all the schools in $S^*$ and let $s^* \in S^*$ be the last school in $S^*$ that is ranked in $Q_i^*$ (formally $s Q_i^* s^*$ for all $s \in S^*$). Then $S^* \subseteq S$ is a safe set for $t_i$ in $M$ if $M_i(Q_i^*, Q_{-i}) Q_i^* s^*$ for every $Q_{-i}$.

Example 1 (Safe set of schools). For simplicity, the example is presented for DA$^3$ in an environment with 4 schools but it can easily be extended to more schools. The left panel represents the students’ preferences and the right panel represents the schools’ priorities. Each school has one seat and the notation ( ) indicates that the rest of the ordering is arbitrary.
At first glance, it may seem that for $t_4$, ranking $Q_4 : s_1 s_2 s_4$ is undominated. By ranking $s_4$ where $t_4$ has the highest priority, $t_4$ makes sure that if $t_1$ and $t_2$ obtain the unique seats at $s_1$ and $s_2$, $t_4$ would not end up unassigned. But note that if $t_1$ and $t_2$ are assigned to $s_1$ and $s_2$, $t_2$ cannot at the same time be assigned to $s_3$. If $t_2$ is assigned to either $s_1$ or $s_2$, then student $t_4$ has the highest priority at $t_3$ among the remaining students. Thus, for $t_4$, reporting $Q_4 : s_1 s_2 s_3$ dominates reporting $Q_4 : s_1 s_2 s_4$.

In the above terminology, $\{s_1, s_2, s_3\}$ is a safe set for $t_4$ because when ranking $s_1, s_2,$ and $s_3$, student $t_4$ is certain to be assigned to a school she reports (weakly) above the school she reports last in $\{s_1, s_2, s_3\}$ (whatever other students report).

A safe strategy is a strategy that contains a safe set. Among other things, a student who plays a safe strategy is guaranteed to be assigned.

We say that school $s_j$ is safe-if-favorite for $t_i$ if $t_i$ is among the $q_j$-students with highest priority at school $s_j$. The terminology refers to $BOS^k$. If school $s_j$ is safe-if-favorite for $t_i$ and $Q_i(1) = s_j$, then $BOS^k_i(Q_i, Q_{-i}) = s_j$ for all $Q_{-i}$ (but not necessarily if $Q_i(x) = s_j$ for some $x \neq 1$).

**Example 1** (Continued). School $s_3$ is safe-if-favorite for $t_2$ because $t_2$ has the highest priority at $s_3$. For any $Q_2 : s_3$ ( ), we have $BOS^k_2(Q_2, Q_{-2}) = s_3$ for all $Q_{-2}$. For the same reason, both schools $s_1$ and $s_2$ are safe-if-favorite for $t_1$.

Finally, for any mechanism $M$ and strategy $Q_i$, the range of $Q_i$ is the set of all the schools $t_i$ could be assigned to when she reports $Q_i$; that is,

$$Range(Q_i) = \{s \in S \mid M_i(Q_i, Q_{-i}) = s \text{ for some } Q_{-i}\}.$$.

**Example 1** (Continued). Consider $Q_2 : s_1 s_3 s_2$. Note that $s_3$ is safe-if-favorite for $t_2$. Thus, $t_2$ is never assigned to $s_2$ or $t_2$ when reporting $Q_2$ in $DA^3$. Also, there exists $Q_{-2}$ for which $DA^3_2(Q_2, Q_{-2}) = s_1$, for example when $t_1$ does not rank $s_1$. Therefore, $Range(Q_2) = \{s_1, s_3\}$ in $DA^3$.

## 5 Dominant and undominated strategies in $DA^k$

### 5.1 Preliminary results

In this subsection, we present some preliminary results that shed light on the structure of dominant and undominated strategies in $DA^k$. These results are used to prove the characterizations of dominant and undominated strategies in Sections 5.2 and 5.3.

We know from Example 1 that undominated strategies are not as simple as they may seem in $DA^k$. Strictly speaking, it is not even the case that all undominated strategies are clean (see Section 7). For example, when a student has the highest priority at her most preferred school, any strategy in which she ranks her favorite school first is undominated. This includes many unclean
undominated strategies. In these unclean undominated strategies however, the student is always assigned to her first choice and any violation of (i), (ii) or (iii) (in the definition of a clean strategy) never has any impact on the student’s assignment.

Lemma 3 at the end of this section shows that this is true for any unclean undominated strategy. If an undominated strategy violates (i), (ii), or (iii), it must contain a safe set past which the student can inconsequentially rank unacceptable schools, rank schools with switch, or not rank any additional schools.

Lemma 3 relies on the two next lemmas which will also be useful in the characterizations and are interesting in their own right. Two strategies \( Q_i \) and \( Q'_i \) are equivalent if the outcome for \( t_i \) is the same under \( Q_i \) and \( Q'_i \) for all \( Q_{-i} \). Lemma 1 shows that for every unclean undominated strategy there exists an equivalent clean strategy. Lemma 1 is useful when studying the properties of undominated strategies outcomes in DA\(^k\) because it allows us to focus on clean profiles only (i.e., profiles made of clean strategies).

**Lemma 1.** For all \( k \in \mathbb{N} \) and any undominated strategy \( Q_i \) of DA\(^k\), there exists an equivalent strategy \( Q'_i \) of DA\(^k\) that is clean.

Lemma 1 follows from the strategy-proofness of DA (Dubins and Freedman, 1981). Let \( Q'_i \) be any clean strategy of DA\(^k\) which reports first all acceptable schools ranked in \( Q_i \). By strategy-proofness, if \( Q'_i \) were \( t_i \)'s true preferences, \( t_i \) would always (weakly) prefer ranking \( Q'_i \) to ranking \( Q_i \). But by construction of \( Q'_i \), this implies that \( t_i \) also likes the outcome of \( Q'_i \) weakly better than the outcome of \( Q_i \) with respect to \( t_i \)'s true preferences \( R_i \). Because \( Q_i \) is undominated, the only way this can be true is if \( Q_i \) and \( Q'_i \) always yield the same outcome.

Intuitively, Lemma 2 shows that, in DA\(^k\), a student will regret “wasting” ranked schools unless she is absolutely sure that this “waste” will never have any impact on her assignment. For instance, consider the unsafe strategy \( Q_i : R_i(1) R_i(3) \preceq t_i \) in DA\(^4\) for some unacceptable school \( s \). Lemma 2 implies that \( t_i \) will regret not having ranked \( R_i(2) \) for some subprofile \( Q^*_{x_i} \). By this we mean that when others report \( Q^*_{x_i} \) and \( t_i \) reports \( Q_i \), \( t_i \) will end up assigned to \( R_i(3) \) and could have been assigned to \( R_i(2) \) if she had reported \( Q'_i : R_i(1) R_i(2) R_i(3) s \) instead. Lemma 2 also implies that when \( t_i \) reports the unsafe \( Q'_i \), she will regret not having ranked an acceptable school instead of \( s \) for some subprofiles \( Q^*_{x_i} \) (assuming \( t_i \) has at least four acceptable schools).

The lemma is based on the concept of an accessible school. A school \( s^* \) is accessible for \( t_i \) given \( Q_{-i} \) if \( t_i \) is certain to be assigned to a school she ranks at least as high as \( s^* \) when ranking \( s^* \). Formally, school \( s^* \) is accessible for \( t_i \) in mechanism \( M \) given \( Q_{-i} \) if for all \( Q_i \) that ranks \( s^* \), we have \( M_i(Q_i, Q_{-i}) Q_i s^* \). Lemma 2 shows that if \( t_i \) is assigned to \( s \) under some profile \( Q \), then any school \( s^* \) that \( Q_i \) ranks lower than \( s \) or that is not ranked under \( Q_i \) is accessible under some other subprofile \( Q^*_{x_i} \) for which \( t_i \) is still assigned to \( s \) when reporting \( Q_i \).

**Lemma 2** (Regret if waste ranked school).

For any \( k \in \mathbb{N} \), \( Q_i, t_i \in T \), and \( s \in S \cup \{t_i\} \), suppose that DA\(^k\)(\( Q_i, Q_{-i} \)) = \( s \)

---

8 Indeed, by Lemma 1, if all outcomes of clean undominated strategies in DA\(^k\) satisfy property \( X \), then all undominated strategy outcomes in DA\(^k\) satisfy property \( X \).

9 By which we mean that \( t_i \)'s preferences would take the form \( R_i : Q'_i \).
Proposition 1

(Proposition in the set of dominant strategies in DA)

Equipped with the lemmas from the last subsection, we can now characterize 5.2 Characterizing dominant strategies

(b) and (c) can never have any impact on $t_1$ can be true is if there exists a safe set and if that safe set satisfies (i) and (ii).

(c) adding unranked acceptable schools. Because $Q_i$ from $Q_i$ equivalent we can construct an

the strategy must fail to rank some acceptable school $\hat{s}$ unacceptable schools. For such a dominant strategy to violate both (i) and (ii),

the necessity of (i) and (ii) for dominant strategies with no switches and no

DA of $s$.

Lemma 3. For all $k \in \mathbb{N}$, if $Q_i$ is an unclean undominated strategy in $DA^k$, then there exists a safe set $S^S$ ranked in $Q_i$ such that

(i) any schools that are switched in $Q_i$ are ranked after $S^S$, and

(ii) any unacceptable school is ranked after $S^S$.

When $Q_i$ is an unclean undominated strategy, we know from Lemma 1 that we can construct an equivalent clean strategy $Q'_i$. Strategy $Q'_i$ is constructed from $Q_i$ by (a) switch alternatives in $Q_i$, (b) deleting unacceptable schools, and (c) adding unranked acceptable schools. Because $Q_i$ and $Q'_i$ are equivalent, (a), (b) and (c) can never have any impact on $t_i$'s assignment. The only way this can be true is if there exists a safe set and if that safe set satisfies (i) and (ii).

5.2 Characterizing dominant strategies

Equipped with the lemmas from the last subsection, we can now characterize the set of dominant strategies in $DA^k$.

Proposition 1 (Dominant strategies in $DA^k$).

For all $k \in \mathbb{N}$, strategy $Q_i$ is dominant in $DA^k$ if and only if either

(i) all the acceptable schools in $R_i$ are ranked without switches in $Q_i$ and these are the only schools ranked in $Q_i$, i.e., $Q_i : R_i(1) \ldots R_i(\#R_i)$ $t_i$, or

(ii) for some $q \leq \min\{k, \#R_i\}$, the $q$ most preferred schools in $R_i$ form a safe set that is ranked first in $Q_i$ and there is no switch among those $q$ schools in $Q_i$, i.e., $Q_i : R_i(1) \ldots R_i(q)$ ( ).

The sufficiency of (i) and (ii) follows directly from the strategy-proofness of DA (Dubins and Freedman, 1981). By Lemma 3, it is sufficient to prove the necessity of (i) and (ii) for dominant strategies with no switches and no unacceptable schools. For such a dominant strategy to violate both (i) and (ii), the strategy must fail to rank some acceptable school $\hat{s}$. This means that $t_i$ will

10 Lemma 2 is used to guarantee this last point. If there is no safe set satisfying (i) and (ii), Lemma 2 can be used to show that for some $Q^*_s$, a school ranked before school $DA^k(Q_i, Q^*_s)$ in strategy $Q'_i$ is accessible and that $DA^k(Q'_i, Q^*_s)$ must therefore differ from $DA^k(Q_i, Q^*_s)$, contradicting the fact that $Q_i$ and $Q'_i$ are equivalent.
sometimes be unassigned, or assigned to a school \( s^* \) that \( t_i \) likes less than \( \hat{s} \). But in both these cases, Lemma 2 tells us that for some sub-profile \( Q_{i-}^* \), \( t_i \) could have been assigned to \( \hat{s} \) if \( t_i \) had reported different preferences, which makes it impossible for \( Q_i \) to be a dominant strategy.

5.2.1 More dominant strategies as \( k \) increases and under the single tie-breaking rule: evidence from simulations

Proposition 1 notably implies that the proportion of agents who have a dominant strategy increases with \( k \) (for a given profile of preferences, priorities, and capacities). First, as \( k \) increases, more and more students can rank all their acceptable schools (condition (i)).

Martin: We already use \( q \) for capacities, we should use a different letter here

But more importantly, the number of students who have a safe set covering their \( q \leq k \) preferred schools weakly increases with \( k \) (condition (ii)). Indeed, a set of schools \( S \) is safe for \( t_i \) if the set of students who have a higher priority than \( t_i \) at any of the schools in \( S \) contains less students that the total number of seats at schools in \( S \).

Oftentimes, this number is in fact strictly increasing, in particular if priorities are correlated across schools. Such correlations can occur when priorities are determined by test scores. Correlations can also result from the tie-breaking rule in use. In many school districts, the criteria used to determine students’ priorities are not sufficient to generate a strict priority ranking of the students. To apply \( DA^k \), ties in priorities are therefore broken, usually at random.

The two most common tie-breaking rules are the single tie-breaking rule (STB) and the multiple tie-breaking rule (MTB). STB breaks ties in the same way at every school whereas MTB uses a different random order to break ties at each school.

The literature suggests that STB performs better than MTB in terms of efficiency (see the references in Footnote 5). Condition (ii) in Proposition 1 indicates that STB also induces better incentives than MTB as measured by the proportion of students with a dominant strategy. We illustrate this last point through simulations in Figure 1. In the figure, the proportion of students with a dominant strategy is always higher under STB than under MTB.\(^{11}\)

The next corollary of Proposition 1 shows that correlations in priorities also have implications for stability.

**Corollary 1.** For all \( k \in \mathbb{N} \) and any \( Q_{i-} \), if \( Q_i \) is a dominant strategy in \( DA^k \), then \( t_i \) does not cause instabilities in \( DA^k(Q_i, Q_{i-}) \).

Corollary 1 follows from observing that in both (i) and (ii) in Proposition 1, \( t_i \) applies and is rejected from every school she likes better than her assignment.

When combined with Proposition 1, Corollary 1 implies that the lines in Figure 1 provide an upper bound on the number of students who could cause instabilities. The upper bound corresponds to one minus the proportions in the figures and relies on the assumption that students play dominant strategies.

\(^{11}\) The linearity of the curves is a consequence of averaging across different random profiles. Monotonicity in \( k \) on the other hand is a feature of every individual profile as Proposition 1 shows.
when such strategies are available. The simulation therefore suggests that STB could also induce more *stable* assignments than MTB.
Figure 1: Proportion of students with a dominant strategy in $DA_k$ as a function of the number of schools that students can rank. The value for each $k$ is the average over 100 random profiles of preferences, priorities, and capacities, with 1000 students and 10 schools. In every profile, all 10 schools are acceptable to every student. The four lines correspond to different values of the sum of seats at the 10 schools (the 1000 case is in slight violation of the short-supply assumption). Priorities are constructed by forming 4 random priority groups at each school. Ties inside the groups are then broken according to the multiple tie-breaking rule (subfigure (a)) or the single tie-breaking rule (subfigure (b)).
5.3 Characterizing undominated strategies

We now turn to the characterization of undominated strategies. Clearly, if \( t_i \) has dominant strategies, then these dominant strategies are \( t_i \)'s only undominated strategies. It remains to determine the form of undominated strategies when \( t_i \) does not have a dominant strategy. The next lemma shows that in this case, \( t_i \) must rank \( k \) (acceptable) schools for a strategy to be undominated.

**Lemma 4.** For all \( k \in \mathbb{N} \), if \( Q_i \) is not a dominant strategy in \( DA_k \) and \( \#Q_i < k \), then \( Q_i \) is dominated in \( DA_k \).

The intuition is similar to the previous lemmas. If neither case (i) nor case (ii) in Proposition 1 apply and if \( \#Q_i < k \), then \( t_i \) does not rank one of her acceptable school \( s \). Also, there is a risk for \( t_i \) to be unassigned or to be assigned to a school that she likes less than \( s \). Then, by Lemma 2, the strategy that ranks the acceptable schools in \( Q_i \) as-well as \( s \) without switches dominates \( Q_i \).

By Lemma 4, ranking \( k \) acceptable schools is necessary for \( Q_i \) to be undominated (if \( Q_i \) is not dominant). But as we illustrated in Example 1, it is not sufficient. In the example, strategy \( Q_4 : s_1 s_2 s_4 \) consists of three acceptable schools. It is however dominated in \( DA_3 \) for \( t_4 \) because \( \{s_1, s_2, s_3\} \) is a safe set for \( t_4 \).\(^{12}\)

The problem with \( Q_4 \) is that \( t_4 \) has a safe set that is in some sense “everywhere above” \( Q_4 \). To make this more precise, let us introduce the concept of a **minimal safe set**. For a set of schools \( S^* \subseteq S \), let \( w^{S^*} \) be the worst school in \( S^* \) according to \( R_i \). A minimal safe set \( S^{MS} \) is a safe set for which \( S^{MS} \setminus \{w^{S^{MS}}\} \) is not a safe set. We then have the following characterization.

**Proposition 2** (Undominated strategies in \( DA_k \)).

For all \( k \in \mathbb{N} \), \( Q_i \) is an undominated strategy in \( DA_k \) if and only if either

(i) \( Q_i \) is a dominant strategy in \( DA_k \), or

(ii) \( k \) acceptable schools are ranked without switches in \( Q_i \) and for any minimal safe set \( S^{MS} \) with \( \#S^{MS} \leq k \) and \( S^{MS} \not\in Q_i \),

\[ s P_i w^{S^{MS}} \quad \text{for some } s \in Q_i \text{ with } s \notin S^{MS}. \]

5.3.1 Uncontroversial recommendations in \( DA_k \)

Proposition 2 summarizes all the recommendations that derive from the lemmas of Section 5.1 and from Proposition 1. First, if a student is able to rank all her acceptable schools, doing so is optimal (point (i) and Proposition 1).

Second, as in \( DA \), unclean strategies are never optimal. By Proposition 2, the only case in which an undominated strategy is unclean corresponds to (ii) in Proposition 1. Even then, playing an unclean strategy is a risk not worth taking. Whatever strategy a student decides to play, (a) she never loses from playing an equivalent clean strategy instead, and (b) she might gain from it if she wrongly believed that case (ii) in Proposition 1 applies. In this sense, no unclean undominated strategy is robust to a misappreciation of the priority structure.

\(^{12}\) It is easy to extend the example in such a way that \( t_4 \) does not have a dominant strategy in \( DA^3 \), for example by adding a school \( s^* \) that \( t_4 \) ranks first but at which she has the lowest priority.
Third, students should pay close attention to the priority structure in deciding which strategy to adopt. In particular, students might be too conservative if they do not take into account the interaction between their priorities at different schools. Ranking “safety” schools at which a student has a high priority may seem wise if the student is worried she might end up unassigned. As case (ii) in Proposition 2 shows however, safety strategies may be dominated by strategies that rank an appropriate combination of more risky schools at which the student has slightly lower priorities (see Example 1).

6 Dominant and undominated strategies in $BOS^k$

6.1 Preliminary results

Again, we start with some preliminary results about the form of dominant and undominated strategies in $BOS^k$. Contrary to $DA^k$, undominated strategies in $BOS^k$ feature non-trivial switches, i.e., switches that do affect students’ assignments. This is a consequence of $BOS$ being manipulable (as opposed to $DA$).

Despite allowing for non-trivial switches, undominated strategy in $BOS^k$ share some common properties with the undominated strategies of $DA^k$.

**Lemma 5.** For all $k \in \mathbb{N}$, if strategy $Q_i$ is undominated in $BOS^k$, then $Range(Q_i)$ contains only acceptable schools.$^{13}$

Although all schools in the range of an undominated strategy $Q_i$ are acceptable, not all schools in $Q_i$ belong to the range of $Q_i$ in $BOS^k$. As in $DA^k$, some strategies of $BOS^k$ are safe and contain ranked schools that never affect the final assignment. These safe strategies are much more rare in $BOS^k$ than in $DA^k$ however, as the next lemma shows.

Let an over-supplied set of schools $O \subseteq S$ be a set of school for which $\sum_{s_j \in O} q_j \geq n$.

**Lemma 6** (Safe strategies in $BOS^k$).
For all $k \in \mathbb{N}$, a strategy $Q_i$ is safe in $BOS^k$ if and only if

(i) $Q_i(1)$ is safe-if-favorite, or

(ii) there exists an over-supplied set of schools $O \subseteq Q_i$ with $\#O \leq k$.

Note that (ii) is ruled out by the short-supply assumption. Under the short-supply assumption, $BOS^k$ rarely features safe strategies because when $Q_i(1)$ is not safe-if-favorite, a student can be rejected in the first round while all the schools are filled in the first round. When this occurs, $t_i$ ends up unassigned because $BOS^k$ does not allow $t_i$ to claim the seats of students who were assigned in the first round in later rounds of $BOS^k$.

Combined with Lemma 6, Lemma 7 shows that only safe strategies can dominate a strategy that contains $\min\{k, \#R_i\}$ acceptable schools.

---

13 The lemma is intuitive but the construction of a strategy $Q'_i$ dominating $Q_i$ when $Range(Q_i)$ contains unacceptable schools is not trivial (see Appendix C). In particular, it is not enough to simply replace the unacceptable school in $Q_i$ by the preferred acceptable school not ranked in $Q_i$. 

---
Lemma 7. For all $k \in \mathbb{N}$, if $Q'_i$ ranks $\min\{k, \#R_i\}$ schools all of which are acceptable and if $Q_i$ dominates $Q'_i$, then $Q_i$ is safe.

Intuitively, when $Q_i$ is unsafe and different from $Q'_i$, it is always possible to construct a sub-profile $Q'^*_i$ for which $t_i$ is assigned when playing $Q'_i$ but is unassigned when playing $Q_i$. Then the lemma follows from the fact that $Q'_i$ only ranks acceptable schools.

6.2 Characterizing dominant strategies

As the preliminary results suggest, $BOS^k$ rarely features dominant strategies.

Proposition 3 (Dominant strategies in $BOS^k$).
For all $k \in \mathbb{N}$, a strategy $Q_i$ is a dominant strategy in $BOS^k$ if and only if $Q_i(1) = R_i(1)$ and either

(i) $R_i(1)$ is safe-if-favorite, or
(ii) $\#R_i = \#Q_i = 1$.

Intuitively, if $R_i(1)$ is not safe-if-favorite, it is always possible to find another undominated strategy $Q'_i \neq Q_i$ containing $\min\{k, \#R_i\}$ acceptable schools. Then by Lemma 7, $Q_i$ does not dominate $Q'_i$.

From Proposition 3 it is easy to see that, as in $DA^k$, students who have dominant strategies in $BOS^k$ do not cause instabilities.

Corollary 2. For all $k \in \mathbb{N}$ and any $Q_{-i}$, if $Q_i$ is a dominant strategy in $BOS^k$ then $t_i$ does not cause instabilities in $BOS^k(Q_i, Q_{-i})$.

Observe however that the number of students who have a dominant strategy is typically small and does not increase with $k$ in $BOS^k$ (under the short-supply assumption). Thus, the trends illustrated in Figure 1 do not occur in $BOS^k$. In particular, the choice of the tie-breaking rule does not impact the number of students who have a dominant strategy in $BOS^k$. As a consequence, the channel through which stability increases when $k$ increases in $DA^k$ does not play out in $BOS^k$ either (despite Corollary 2).

6.3 Characterizing undominated strategies

Revealing preferences truthfully is a dominant strategy in $DA$. Advising students to do so is therefore uncontroversial. In this sense, $DA$ has better strategic properties than $BOS$ because truthful report is not a dominant strategy in $BOS$. A similar result holds for $DA^k$ and $BOS^k$. As we show in Corollary 3 (Section 7), it is easier to uncontroversially rule out some strategies in $DA^k$ than in $BOS^k$.

The next proposition is instrumental in proving Corollary 3. It shows that the set of undominated strategies of $BOS^k$ is quite large.

---

14 Even with sets of over-supplied schools, the proportion of players with a dominant strategy increases slower under $BOS^k$ than under $DA^k$. This is because in $BOS^k$ safe sets appear as $k$ increases due only to over-supplied set of schools. In addition to these safe sets, $DA^k$ features other safe sets that result from the ability to claim priorities in every round of the assignment procedure.
Proposition 4 (Undominated strategies in $BOS^k$).
For all $k \in \mathbb{N}$, $Q_i$ is an undominated strategy in $BOS^k$ if and only if either

(i) $Q_i(1)$ is $t_i$’s preferred safe-if-favorite and acceptable school, or

(ii) $Q_i(1)$ is not safe-if-favorite and $Q_i$ contains $\min\{k, \#R_i\}$ acceptable schools, one of which $t_i$ prefers to all of her safe-if-favorite and acceptable schools.

Observe that if $R_i(1)$ is safe-if-favorite, (ii) cannot occur and the undominated strategies are those dominant strategies $Q_i$ for which $Q_i(1) = R_i(1)$. Observe also that (ii) does not restrict the ordering of schools in $Q_i$ to match $R_i$ and allows for many non-trivial switches.

To see why (ii) is sufficient recall that, by Lemma 7, when $Q_i$ ranks $\min\{k, \#R_i\}$ acceptable schools, $Q_i$ is not dominated by any unsafe strategy. A safe strategy $Q'_i$ can still dominate $Q_i$. By Lemma 6, such safe strategy $Q'_i$ must always result in the assignment of $t_i$ to her safe-if-favorite school $Q'_i(1)$. But by (ii), $Q_i$ contains some school $s^*$ that $t_i$ prefers to any of her acceptable safe-if-favorite schools, including $Q'_i(1)$. Thus, there will be some sub-profile $Q^*_{-i}$ for which $t_i$ is assigned to $s^*$ when reporting $Q_i$ and to $Q'_i(1)$ when reporting $Q'_i$ and $Q'_i$ cannot dominate $Q_i$.

7 Comparing the manipulability of $DA^k$ and $BOS^k$

Many approaches allow to compare the manipulability of two mechanisms that both fail to be strategy-proof.\textsuperscript{15} In this last section, we propose a new approach for manipulability comparisons and use our characterization of undominated strategies to apply this approach to $DA^k$ and $BOS^k$ (see Corollary 3). We also use our characterization of dominant strategies to apply the criterion developed by Arribillaga and Massó (2015) to $DA^k$ and $BOS^k$.

7.1 Comparing uncontroversial recommendations

As discussed in the Introduction, advising students not to play dominated strategies is uncontroversial because this advice does not depend on any assumption about other students’ strategies. Therefore, when it comes to comparing the manipulability of mechanisms, it seems natural to prefer a mechanism in which dominance excludes a larger set of strategies.

Strictly speaking, the set of undominated strategies of $DA^k$ is not always included into the set of undominated strategies of $BOS^k$. This is because, when the only undominated strategies are dominant strategies in $DA^k$, by case (ii) in Proposition 1, these strategies may rank less than $\min\{k, \#R_i\}$ acceptable schools.

However, as discussed before, when a student has an undominated strategy that ranks less than $\min\{k, \#R_i\}$ acceptable schools, the student cannot lose from reporting a clean undominated strategy instead. If it is granted that playing a clean undominated strategies in $DA^k$ and an undominated strategy in $BOS^k$ are uncontroversial recommendations, then the set of strategies that

\textsuperscript{15} Besides the papers already cited, see e.g. Aleskerov and Kurbanov (1999), Maus et al. (2007a), Maus et al. (2007b), Andersson et al. (2014b), Andersson et al. (2014a) and Fujinaka and Wakayama (2015)).
can be ruled out by uncontroversial recommendations in \( DA^k \) is a superset of that in \( BOS^k \).

**Corollary 3.** For all \( k \in \mathbb{N}, t_i \in T, R_i, \) and \( Q_i \), if \( Q_i \) is a clean undominated strategy in \( DA^k \), then \( Q_i \) is also an undominated strategy in \( BOS^k \). The converse is not true if \( k \geq 2 \).

### 7.2 Comparing dominant strategies

Arribillaga and Massó (2015) introduce another approach to comparing the manipulability of mechanisms. They argue that a mechanism is less manipulable the more it provides students with a truthful dominant strategy. Mechanism \( B \) is at least as manipulable as mechanism \( A \) (in the sense of Arribillaga and Massó (2015)) if for all \( t_i \), whenever \( t_i \) has a truthful dominant strategy given \( R_i \) in \( B \), \( t_i \) also has a truthful dominant strategy given \( R_i \) in \( A \). Formally, for every profile of priorities \( F \) and capacities \( q \) and for all \( t_i \in T \),

\[
\{R_i \mid t_i \text{ has a truthful dominant strategy in } B\} \subseteq \{R_i \mid t_i \text{ has a truthful dominant strategy in } A\}. \tag{1}
\]

In the context of constrained school choice mechanisms \( M^k \), we consider that any strategy in which \( t_i \) reports her min\( \{k, \#R_i\} \) most preferred schools without switches is a truthful strategy.

Mechanism \( B \) is more manipulable than mechanism \( A \) if \( B \) is at least as manipulable as \( B \) but the converse is not true. In the context of constrained school choice mechanisms, this means that there exists a profile of priorities \( F \) and a profile of capacities \( q \) such that \( \subset \) replaces \( \subseteq \) in (1). Mechanism \( B \) is equally manipulable as \( A \) if \( A \) is at least as manipulable as \( B \) and \( B \) is at least as manipulable as \( A \), i.e., \( = \) replaces \( \subset \) in (1) (for all \( F \) and \( q \)).

The following result is a direct corollary of our characterization of dominant strategies in Propositions 1 and 3.

**Corollary 4.** For all \( k \geq 2 \), \( BOS^k \) is more manipulable than \( DA^k \).\(^{16}\)

Propositions 1 and 3 also enable the comparison of \( DA^k \) and \( BOS^k \) for different values of \( k \).

**Corollary 5.** For all \( k \leq m - 1 \), \( DA^{k+1} \) is less manipulable than \( DA^k \).

**Corollary 6.** For all \( k \in \mathbb{N}, BOS^{k+1} \) is equally manipulable as \( BOS^k \).

Figure 2 summarizes Corollaries 4 to 6. The corollaries in this section indicate that the advantage of \( DA \) over \( BOS \) in terms of manipulability carries over to \( DA^k \) and \( BOS^k \).

### 8 Conclusion

Corollary 4 confirms Proposition 3 in Pathak and Sönmez (2013) which shows that \( DA^k \) is less manipulable than \( BOS^k \) using a different criterion based on

\(^{16}\) For \( k = 1 \), \( DA^k \) and \( BOS^k \) are strategically equivalent and the two mechanisms are therefore equally manipulable.
\[ DA > DA^{m-1} > \cdots > DA^1 = BOS = BOS^{m-1} = \cdots = BOS^1 \]

Figure 2: Manipulability comparisons of $BOS^k$ and $DA^k$ in the sense of Arribillaga and Massó (2015), where $A > B$ indicates that $A$ is less manipulable than $B$ and $A = B$ indicates that $A$ and $B$ are equally manipulable.

the nestedness of profiles that admit a truthful Nash equilibrium. Pathak and Sönmez (2013, Section 3) introduce yet another criterion based on the inclusion of profiles at which no player can manipulate. Although comparisons in the sense of Pathak and Sönmez (2013, Section 3) imply comparisons in the sense of Arribillaga and Massó (2015), the converse is not true.\(^{17}\) In particular, none of the results summarized in Figure 2 have any implication for comparisons in the sense of Pathak and Sönmez (2013, Section 3). Whether constrained school choice mechanisms can be compared using Pathak and Sönmez (2013, Section 3) and how these comparisons would play out is an open question.

Acknowledgments

We are grateful to Estelle Cantillon, Caterina Calsamiglia, Eun Jeong Heo, François Maniquet and John Weymark for helpful discussions and comments. We also thank Tommy Andersson, Paul Edelman, Jordi Massó, William Phan, Yves Sprumont, Myrna Wooders and participants to presentations at Université Catholique de Louvain, Université Saint-Louis, Vanderbilt University, the 9th Conference on Economic Design and the 13th Meeting of the Society for Social Choice and Welfare for useful questions and suggestions. Martin Van der Linden thanks the CEREC and the CORE for their hospitality during visits. Support from the ERC under the European Union’s Seventh Framework Programme (FP/2007–2013)/ERC Grant Agreement No. 269831, the Kirk Dornbush summer research grant, the National Science Foundation grant IIS-1526860 and the Fond National de la Recherche Scientifique (Belgium, mandat d’aspirant FC 95720) is gratefully acknowledged.

Appendix

A Additional notation

We introduce some additional notation and terminology we use in the Appendix.

For reported preferences $Q_i$ and preferences $R_i$, we abuse the notation and write $Q_i = R_i$ to refer to case (i) in Proposition 1, that is $Q_i = R_i$ means that $Q_i$ is of the form $Q_i : R_i(1) \ldots R_i(\#R_i) \ t_i \ldots$. For any two strategies $Q_i$ and $Q_i'$, we also write $Q_i = Q_i'$ if both strategies share the same set of ranked schools and report those schools in the same order. Similarly $Q_i \neq R_i$ means that $Q_i$ is not of the form $Q_i : R_i(1) \ldots R_i(\#R_i) \ t_i \ldots$.

\(^{17}\) See Arribillaga and Massó (2015) and Van der Linden (2016) for a discussion of the differences between the two criteria.
The higher priority of student $t_g$ over $t_h$ at school $s_j$ is denoted $t_g F_j t_h$. Given a strategy $Q_i$, the truncation of $Q_i$ after school $s$ is another strategy $Q'_i$ obtained from $Q_i$ by deleting all schools $s' \in Q_i$ reported after $s$.

**B Proofs for dominant and undominated strategies in $DA^k$**

**B.1 Preliminary results**

**Lemma 1**

Let $Q'_i$ be any strategy of $DA^k$ which ranks exactly $\min\{k, \#R_i\}$ acceptable schools without switch, including all acceptable schools ranked in $Q_i$. For any strategy $\tilde{Q}_i$, let $R^{\tilde{Q}_i}$ be any preference relation over $S \cup \{t_i\}$ of the form

$$R^{\tilde{Q}_i} : \tilde{Q}_i, t_i, Q^{S \setminus \tilde{Q}_i},$$

where $Q^{S \setminus \tilde{Q}_i}$ is any sub-orderings of the schools in $S \setminus \tilde{Q}_i$. Because $DA$ is non-manipulable (Dubins and Freedman, 1981), we have

$$DA_i(Q'_i, Q^m_i) R^{\tilde{Q}_i} DA_i(Q_i, Q^m_i), \quad \text{for all } Q^m_i.$$

In particular,

$$DA_i(Q'_i, Q^R_{-i}) R^{\tilde{Q}_i} DA_i(Q_i, Q^R_{-i}),$$

for all $Q^R_{-i}$, with $\#Q^R_k \leq k$ for all $t_j \in T \setminus \{t_i\}$.

But because $DA^k$ is obtained from $DA$ by considering only the profiles $Q^k$ with $\#Q^k_j \leq k$ for all $t_j \in T$, the last displayed relation implies

$$DA^k_i(Q'_i, Q^R_{-i}) R^{\tilde{Q}_i} DA^k_i(Q_i, Q^R_{-i}), \quad \text{for all } Q^R_{-i}. $$

By construction, $Q'_i$ is without switch, and therefore, the last displayed relation implies

$$DA^k_i(Q'_i, Q^R_{-i}) R_i DA^k_i(Q_i, Q^R_{-i}),$$

for all $Q^R_{-i}$ such that $DA^k_i(Q'_i, Q^R_{-i}), DA^k_i(Q_i, Q^R_{-i}) \in (Q'_i \cup \{t_i\})$. \(2\)

Clearly, by definition of $DA^k$,

$$DA^k_i(Q'_i, Q^R_{-i}) \in (Q'_i \cup \{t_i\}), \quad \text{for all } Q^R_{-i},$$

and (2) simplifies to

$$DA^k_i(Q'_i, Q^R_{-i}) R_i DA^k_i(Q_i, Q^R_{-i}),$$

for all $Q^R_{-i}$ such that $DA^k_i(Q_i, Q^R_{-i}) \in (Q'_i \cup \{t_i\})$. \(3\)

Now, because every acceptable school ranked in $Q_i$ is ranked in $Q'_i$, the only cases (if any) in which $DA^k_i(Q_i, Q^R_{-i}) \notin (Q'_i \cup \{t_i\})$ is when $t_i F_i DA^k_i(Q_i, Q^R_{-i})$. But because $Q'_i$ only ranks acceptable schools, $DA^k_i(Q'_i, Q^R_{-i}) R_i t_i$ for all $Q_{-i}$ and therefore, in these cases too, $DA^k_i(Q'_i, Q^R_{-i}) R_i DA^k_i(Q_i, Q^R_{-i})$.  

19
Thus, (3) further simplifies to
\[ \text{DA}^k_i(Q', Q_{-i}^*) R_i \text{ DA}^k_i(Q, Q_{-i}^*), \quad \text{for all } Q_{-i}^*. \tag{4} \]

Then, if in addition of (4)
\[ \text{DA}^k_i(Q', \tilde{Q}_{-i}) P_i \text{ DA}^k_i(Q, \tilde{Q}_{-i}), \quad \text{for some } \tilde{Q}_{-i}, \]

\(Q'_i\) would weakly dominate \(Q_i\). But this would contradict the assumption that \(Q_i\) is undominated in \(\text{DA}^k\). Therefore, we must in fact have
\[ \text{DA}^k_i(Q, Q_{-i}^*) R_i \text{ DA}^k_i(Q', Q_{-i}^*), \quad \text{for all } Q_{-i}^*. \tag{5} \]

But because \(R\) is antisymmetric, (4) and (5) imply
\[ \text{DA}^k_i(Q', Q_{-i}^*) = \text{DA}^k_i(Q, Q_{-i}^*), \quad \text{for all } Q_{-i}^*, \]

the desired result.

**Lemma 2**

Let \(B\) be the set of schools that \(t_i\) ranks above \(\hat{s}\) in \(Q_i\). These are the schools \(t_i\) applied to in the course of \(\text{DA}^k\) under \((Q_i, Q_{-i})\), but did not get assigned to. Because \(t_i\) was rejected from the schools in \(B\), it must be that, in the list of assignments \(\text{DA}^k(Q_i, Q_{-i})\), there is another student assigned to each of the available seats in each of the schools in \(B\). Let this set of students be \(A \subset T\).

Now construct \(Q_{-i}^*\) as follows :

- For all \(t_j \in A\), let \(Q_j^*\) be the strategy in which \(t_j\) ranks only \(\text{DA}^k_i(Q, Q_{-i})\).
- For all \(t_h \in T \setminus \{A \cup \{t_i\}\}\), let \(Q_h^*\) be any strategy in which \(t_h\) ranks neither \(s^*\) nor \(\hat{s}\).

By construction, for every school \(s_j \in B\), there are at least \(q_j\)-students with higher priority at \(s_j\) than \(t_i\) who rank \(s\) first in \(Q_{-i}^*\). Thus, \(t_i\) will be rejected from any of these schools over the course of \(\text{DA}^k\) given that the reported profile is \((Q_i, Q_{-i}^*)\). Therefore, \(\text{DA}^k_i(Q, Q_{-i}^*) = \hat{s}\) implies
\[ \hat{s} Q_i \text{ DA}^k_i(Q, Q_{-i}^*). \]

By construction again, no students rank \(\hat{s}\) in \(Q_{-i}^*\). Therefore, \(\text{DA}^k_i(Q, Q_{-i}^*) = \hat{s}\) implies
\[ \text{DA}^k_i(Q, Q_{-i}^*) Q_i \hat{s}. \]

Because \(Q_i\) is antisymmetric, the last two displayed relations imply
\[ \text{DA}^k_i(Q, Q_{-i}^*) = \hat{s}, \]

which proves (i).

Again, by construction, no-one applies to \(s^*\). Thus, for any \(Q_i^*\) with \(s^* \in Q_i^*\), we have
\[ \text{DA}^k_i(Q_i^*, Q_{-i}^*) Q_i^* s^*, \]

which proves (ii).
Lemma 3

The proof is by contradiction. Lemma 1 tells us that there exists a strategy $Q'_i$ which ranks $\min\{k, \#R_i\}$ acceptable schools without switch including all the acceptable schools in $Q_i$ and which satisfies

$$DA^k_i(Q'_i, Q_{-i}) = DA^k_i(Q_i, Q_{-i}), \quad \text{for all } Q_{-i}. \quad (6)$$

We break the proof down in three cases which correspond to the three possible sources of uncleanness (the labeling of the sources of uncleanness matches the labeling in the text).

(ii) unacceptable schools are ranked in $Q_i$.

Let $\not x$ be any unacceptable school ranked in $Q_i$. By (6) and because only acceptable schools are ranked in $Q'_i$, $\not x \notin \text{Range}(Q_i)$. But this implies $\not x$ is ranked after a safe set, proving that $Q_i$ contains a safe set and that (2) holds.

(i) $Q_i$ contains switches.

Take any schools $s$ and $s'$ which are switched in $Q_i$. Without loss of generality, let $s Q_i s'$ and $s' P_i s$. In order to derive a contradiction, assume that either $s$ or $s'$ are not ranked after a safe set in $Q_i$ (potentially because $Q'_i$ does not contain a safe set).

Since $s Q_i s'$ we have $s \in \text{Range}(Q_i)$. By point (ii) this means that $s$ is an acceptable school of $Q_i$. Because $s' P_i s$, $s'$ is also an acceptable school ranked in $Q_i$. Hence, by construction of $Q'_i$, both $s$ and $s'$ are ranked in $Q'_i$.

By Lemma 2, there exists $Q^*_i$ such that $DA^k_i(Q_i, Q^*_i) = s$ and $s'$ is accessible. But then $s' \in Q'_i$ implies

$$DA^k_i(Q'_i, Q^*_i) \not x s', Q'_i s,$$

and hence $DA^k_i(Q'_i, Q^*_i) \not s = DA^k_i(Q_i, Q^*_i)$, contradicting (6).

(iii) $Q_i$ contains less than $\min\{k, \#R_i\}$ acceptable schools.

Let $s'$ be any acceptable school ranked in $Q'_i$ but not in $Q_i$. In order to derive a contradiction, assume that $Q_i$ contains no safe set. The absence of safe set ranked in $Q_i$ implies that $DA^k_i(Q_i, Q_{-i}) = t_i$ for some $Q_{-i}$. Then by Lemma 2, there exists $Q^*_i$ such that $DA^k_i(Q_i, Q^*_i) = t_i$ and $s'$ is available. But then

$$DA^k_i(Q'_i, Q^*_i) Q'_i s' \not t_i,$$

which implies $DA^k_i(Q'_i, Q^*_i) \not DA^k_i(Q_i, Q^*_i)$ contradicting (6).

B.2 Characterizing dominant strategies

Proposition 1

 Sufficiency.

(i) The sufficiency of (i) follows directly from the fact that $DA$ is non-manipulable. (Dubins and Freedman, 1981). See the beginning of the proof of Lemma 1 for an explanation of how this extends to $DA^k$. 

21
(ii) By construction, strategy $Q_i$ ranks the $q$ most preferred schools in $R_i$ without switches. Following the same argument as in Lemma 1,

$$DA_k^k(Q_i, Q_{-i}) R_i DA_k^k(Q_i', Q_{-i}), \quad \text{for all } Q_i' \text{ and all } Q_{-i}, \text{ such that } DA_k^k(Q_i, Q_{-i}) R_i R_i(q).$$

But by definition of a safe set and because $Q_i$ is without switches, if $R_i(q) R_i DA_k^k(Q_i', Q_{-i}),$ we also have

$$DA_k^k(Q_i, Q_{-i}) R_i R_i(q) R_i DA_k^k(Q_i', Q_{-i}).$$

Hence $DA_k^k(Q_i, Q_{-i}) R_i DA_k^k(Q_i', Q_{-i})$ holds for all $Q_i'$ and all $Q_{-i},$ and we are done.

**Necessity.**
The proof is by contradiction. Assume that there exists a dominant strategy $Q_i$ for which neither (i) nor (ii) hold.

Because $Q_i$ is a dominant strategy, $Q_i$ is also an undominated strategy. This means Lemma 3 applies, and there exists a clean strategy $Q_i'$ which is equivalent to $Q_i,$ and is therefore also a dominant strategy.

**Case 1:** $Q_i' \neq R_i.$

Because $Q_i'$ is without switches, without unacceptable schools, and $\min\{k, \#R_i\}$ schools are ranked in $Q_i'$, there is an acceptable school $s^*$ which is not ranked in $Q_i'$ and which is such that $s^* P_i w^{Q_i'}.$ By construction of $Q_i'$ and because $Q_i$ violates both (i) and (ii), $w^{Q_i'}$ is not ranked after any safe set in $Q_i'$ and hence $w^{Q_i'} \in \text{Range}(Q_i').$ But then by Lemma 2, there is a subprofile $Q_{-i}^*$ and a strategy without switch $Q_i^*$ such that $DA_k^k(Q_i', Q_{-i}^*) = w^{Q_i'}$ and $DA_k^k(Q_i^*, Q_{-i}^*) R_i s^*$, contradicting the fact that $Q_i'$ is a dominant strategy.

**Case 2:** $Q_i' = R_i.$

Because $Q_i'$ is without switch, $Q_i'$ cannot be a safe strategy (otherwise (ii) would hold for $Q_i,$ by construction of $Q_i'$ from $Q_i'$). By Lemma 2 again, if there is an acceptable school $s^* \notin Q_i'$, there is a subprofile $Q_{-i}^*$ and a strategy $Q_i^*$ such that $DA_k^k(Q_i', Q_{-i}^*) = t_i$ and $DA_k^k(Q_i^*, Q_{-i}^*) = s^*$, contradicting the fact that $Q_i'$ is a dominant strategy. Therefore, all acceptable schools are ranked first in $Q_i'$ without switch, and those are the only schools ranked in $Q_i'.$ But because $Q_i$ is unsafe, $Q_i$ is unsafe too, which means no unacceptable school can be ranked in $Q_i.$ Thus, it is also the case that all acceptable schools are ranked first in $Q_i$ without switch, and that those are the only schools ranked in $Q_i,$ contradicting (i).

**Corollary 1**

It is easy to see that if for some $Q_{-i},$ $t_i$ is assigned to an unacceptable school or prefers a school $s$ with an available seat to her assignment, strategy $Q_i$ cannot be dominant. Respectively, ranking no acceptable school or ranking $s$ as the only acceptable school dominates $Q_i$ given $Q_{-i}$.

To see that $t_i$ is not involved in a blocking pair, take any school $s_y$ such that $s_j P_i DA_k^k(Q_i, Q_{-i}).$ By Proposition 1, $s_j$ is an acceptable school and $t_i$ applied and was rejected from $s_j.$ At the round at which $t_i$ was rejected from $s_j,$ there
are $q_j$ students $t_g \neq t_i$ assigned to $s_j$ with higher priority at $s_j$ than $t_i$. If any student $t_h$ is rejected from $s_j$ in a later round of $DA^k$, the seat in $s_j$ previously occupied by $t_h$ is assigned to another student $t_i$ with higher priority at $s_j$ than $t_h$, and hence with higher priority at $s_j$ than $t_i$. Therefore, there cannot be any student $t_h$ with $t_i \in F_j \setminus t_h$ such that $DA^k_h(Q) = s_j$. This means $t_i$ cannot be in a blocking pair with $s_j$.

\section*{B.3 Characterizing undominated strategies}

\textbf{Lemma 4}

By Proposition 1, because $Q_i$ is not dominant, we have $Q_i \neq R_i$ and the $q$ most preferred schools of $Q_i$ do not form a safe set covering the $q$ most preferred schools of $R_i$ for any $q \leq k$. Thus, either

(i) $Q_i$ contains a safe set but there is an acceptable school $s \in R_i$ with $s \notin Q_i$ such that $s P_i w_{\text{Range}(Q_i)}$,

(ii) $Q_i$ is unsafe.

We first show by contradiction that $Q_i$ is unsafe. Assume to the contrary that $Q_i$ is clean. As $\#Q_i < k$, if $Q_i$ is clean, then $\#Q_i = \#R_i$. Thus, either $Q_i$ contains switches contradicting the definition of a clean strategy, or $Q_i$ contains no switches and $Q_i = R_i$, which contradicts the assumption that $Q_i$ is not dominant.

By Lemma 3, because $Q_i$ is an undominated strategy, $Q_i$ contains a safe set $S^g$, which rules out case (i) and we need only consider case (ii). By Lemma 3 again, the safe set $S^g$ contained in $Q_i$ is such that any schools ($s, s'$) that are switched in $Q_i$ are ranked after $S^g$.

Now, the strategy $Q'_i$ that ranks all the schools in $\text{Range}(Q_i)$ as well as $s$, without switch. Because $\#Q_i < k$, $Q'_i$ is a well-defined strategy for $DA^k$.

By an argument similar to the one used in the proof of Lemma 1, $DA^k$ is strategy-proof for any agent with no more than $k$ acceptable schools. Therefore,

$$DA^k_i(Q'_i, Q_{-i}) \leq DA^k_i(Q_i, Q_{-i}),$$

for all $Q_{-i}$.

But because $Q'_i$ is without switch on $\text{Range}(Q_i) \cup \{s\}$ and on $\text{Range}(Q'_i)$, we have

$$DA^k_i(Q'_i, Q_{-i}) \sqsubseteq R_i DA^k_i(Q_i, Q_{-i}),$$

for all $Q_{-i}$. \hfill (8)

Now because $s \notin Q_i$, Lemma 2 applies and there exists $Q^*_i$ such that

$$DA^k_i(Q'_i, Q^*_i) = s P_i w_{\text{Range}(Q_i)} = DA^k_i(Q_i, Q^*_i),$$

which together with (8) implies that $Q'_i$ dominates $Q_i$, the desired result.

\textbf{Proposition 2}

\textbf{Necessity.}

Let $Q_i$ be any undominated strategy of $DA^k$. We show that if $Q_i$ is not a dominant strategy in $DA^k$, then (ii) holds. First, we show that $\#\text{Range}(Q_i) = k$ for any $Q_i$ that is undominated but not dominant in $DA^k$.  

23
Clearly we cannot have \( \#\text{Range}(Q_i) > k \) as that would mean \( Q_i \) is not a well-defined strategy in \( DA^k \). So in order to derive a contradiction, assume that \( \#\text{Range}(Q_i) < k \). This means that either (a) \( Q_i \) contains a safe set with less than \( k \) schools, or (b) \( \#Q_i < k \). But by Lemma 4 and the fact that \( Q_i \) is undominated but not dominant, (b) yields a contradiction. Thus, (a) must hold. Now, consider the strategy \( Q'_i \) constructed from \( Q_i \) by removing a ranked school ranked after the safe set. Strategy \( Q_i \) is equivalent to \( Q'_i \). Clearly, \( \#Q'_i < k \) and because \( Q'_i \) is equivalent to \( Q_i \), \( Q'_i \) is also undominated but not dominant. Again, this contradicts Lemma 4 and hence \( \#\text{Range}(Q_i) = k \).

We now prove each of \( Q_i \)'s properties described in (ii).

**No switch.**

By Lemma 3, no switched schools are in \( \text{Range}(Q_i) \). Since \( \#\text{Range}(Q_i) = k \) for any \( Q_i \) that is undominated but not dominant in \( DA^k \), \( Q_i \) contains no switches.

**\( Q_i \) ranks \( k \) acceptable schools**

By Lemma 3, if \( Q_i \) contains any unacceptable school \( s \), then \( s \) is not in \( \text{Range}(Q_i) \). Again, as \( \#\text{Range}(Q_i) = k \), this implies that \( s \) is ranked in \( Q_i \).

**No minimal safe set “dominates” \( Q_i \)**

We prove the contrapositive: if a minimal safe set “dominates” \( Q_i \), then it satisfies the properties described in the statement of the proposition, then \( Q_i \) is dominated.

Let \( MMS \) be some minimal safe set of schools with \( \#S^{MMS} \leq k \) and \( S^{MMS} \not\subseteq Q_i \), such that for all \( s \in Q_i \), with \( s \not\in S^{MMS} \), we have \( w^{S^{MMS}}(s) \). Consider the strategy \( Q'_i \) which consists in only ranking \( S^{MMS} \) without switch. Because \( \#S^{MMS} \leq k \), \( Q'_i \) is a well-defined strategy in \( DA^k \). By an argument we have already used many times, because \( DA^k \) is strategy-proof for every student who has no more than \( \#S^{MMS} \leq k \) acceptable schools, and because \( Q^{S^{MMS}} \) is without switch

\[
DA^k_i(Q_i^{S^{MMS}}, Q_{-i}) R_i DA^k_i(Q_i, Q_{-i}), \quad \text{for all } Q_{-i} \text{ such that } DA^k_i(Q_i, Q_{-i}) \in Q_i^{S^{MMS}} \cup \{t_i\}.
\]

But because \( w^{S^{MMS}} \) \( R_i \) \( s \) for all \( s \in Q_i \) with \( s \not\in S^{MMS} \), the last displayed relation generalizes to

\[
DA^k_i(Q_i^{S^{MMS}}, Q_{-i}) R_i DA^k_i(Q_i, Q_{-i}), \quad \text{for all } Q_{-i}. \tag{9}
\]

Now because \( S^{MMS} \not\subseteq Q_i \), there exists \( s^* \in Q_i^{S^{MMS}} \) such that \( s^* \not\in Q_i \), and by Lemma 2, there exists \( Q^*_i \) such that

\[
DA^k_i(Q_i^{S^{MMS}}, Q^*_i) = s^* \quad R_i \quad w^{S^{MMS}} \quad R_i \quad DA^k_i(Q_i, Q^*_i).
\]

But because \( s^* \not\in Q_i \), \( s^* \neq DA^k_i(Q_i, Q^*_i) \), and because \( R_i \) is antisymmetric, the last relation in fact yields

\[
DA^k_i(Q_i^{S^{MMS}}, Q^*_i) = s^* \quad P_i \quad DA^k_i(Q_i, Q^*_i). \tag{10}
\]

Together, (9) and (10) show that \( Q_i^{S^{MMS}} \) dominates \( Q_i \), the desired result.
Sufficiency.

Clearly, if $Q_i$ is a dominant strategy (case (i)), $Q_i$ is undominated. Thus, assume that $Q_i$ is not a dominant strategy but satisfies (ii). We need to prove that $Q_i$ is undominated. In order to derive a contradiction, assume that there exists a strategy that dominates $Q_i$.

Because the “domination” relation is transitive, because there are a finite number of strategies and because some strategy dominates $Q_i$, there exists an undominated strategy $Q_i''$ that dominates $Q_i$. Also, by Lemma 1, there exists a clean undominated strategy $Q_i'$ that is equivalent to $Q_i''$. In particular, $Q_i'$ dominates $Q_i$ too.

There are two cases.

**Case 1:** $Q_i'$ is unsafe.

Note that if $Q_i$ and $Q_i'$ rank the same schools, because both $Q_i$ and $Q_i'$ are without switch (by (ii) for $Q_i$ and by construction for $Q_i'$), we have $Q_i = Q_i'$. Hence, $Q_i$ and $Q_i'$ cannot dominate $Q_i$.

Thus, suppose that $Q_i'$ does not rank the same schools as $Q_i$, i.e., there exists $s^* \in Q_i$ with $s^* \notin Q_i'$. By (ii), $s^*$ is acceptable. If $s^* \notin \Range(Q_i)$, then $Q_i$ is safe. But a safe strategy $Q_i$ ranking only acceptable schools cannot be dominated by an unsafe strategy $Q_i'$.

Thus, suppose that $s^* \in \Range(Q_i)$. Because $Q_i'$ is unsafe, $DA^k_i(Q_i', Q_{-i}) = t_i$ for some $Q_{-i}$. But then by Lemma 2, there exists $Q_{-i}^*$ such that

$$DA^k_i(Q_i, Q_{-i}^*) = s^* P_i t_i = DA^k_i(Q_i', Q_{-i}^*),$$

contradicting the assumption that $Q_i'$ dominates $Q_i$.

**Case 2:** $Q_i'$ is safe.

As $Q_i'$ contains a safe set, $\Range(Q_i')$ is a safe set. In fact, $\Range(Q_i')$ is a minimal safe set. Indeed, because $Q_i'$ is without switch, if $\Range(Q_i') \setminus w^{\Range(Q_i')}$ is a safe set, then $\Range(Q_i') = \Range(Q_i') \setminus w^{\Range(Q_i')}$, a contradiction.

Thus, by (ii), either (a) $\Range(Q_i') \subseteq Q_i$ or (b) $s^* P_i w^{\Range(Q_i')}$ for some $s^* \in Q_i$ with $s^* \notin \Range(Q_i')$.

**Subcase (a)(1):** (a) holds and $Q_i'$ is a dominant strategy.

Then by Proposition 1, $Q_i$ is dominant too since $Q_i$ is without switches. This contradicts the assumption that $Q_i'$ dominates $Q_i$.

**Subcase (a)(2):** (a) holds and $Q_i'$ is not dominant.

If $Q_i' = Q_i$, we have a direct contradiction. If instead $Q_i' \neq Q_i$, then $\Range(Q_i') \subseteq Q_i$ implies that $\#Q_i' < k$ as both $Q_i$ and $Q_i'$ are without switches. Therefore, Lemma 4 applies and $Q_i'$ is dominated, a contradiction.

**Subcase (b):** (b) holds.

By the definition of the range, $DA^k_i(Q_i', Q_{-i}) = w^{\Range(Q_i')}$ for some $Q_{-i}$. But then, by Lemma 2, there exists $Q_{-i}^*$ such that

$$DA^k_i(Q_i, Q_{-i}^*) = s^* P_i w^{\Range(Q_i')} = DA^k_i(Q_i', Q_{-i}^*),$$

contradicting the assumption that $Q_i'$ dominates $Q_i$.
C Dominant and undominated strategies in $BOS^k$

C.1 Preliminary results

Lemma 5

In order to derive a contradiction, assume that $s \in \text{Range}(Q_i)$ and $s$ is unacceptable. We construct $Q_i'$ dominating $Q_i$ in $BOS^k$, contradicting the assumption that $Q_i$ is an undominated strategy. The construction of $Q_i'$ is step by step:

- Step 1: If $Q_i(1) \in R_i$, then $Q_i'(1) := Q_i(1)$. Else $Q_i'(1) := R_i(1)$.
  -
- Step $\ell$: If $Q_i(\ell) \in R_i$ and $Q_i(\ell)$ is not yet ranked in $Q_i'(h)$ for $h < \ell$, then $Q_i'(\ell) := Q_i(\ell)$. Else $Q_i'(\ell)$ is the preferred school according to $R_i$ that is not yet ranked in $Q_i'(h)$, for $h < \ell$.
  -
- Last step $\ell^*$ is the minimal step such that either $\ell^* = \# \text{Range}(Q_i)$ or all acceptable schools are ranked in $Q_i'$.

We now prove that $Q_i'$ dominates $Q_i$ in $BOS^k$. First, we show by contradiction that for all $Q_{-i}$, we have

$$BOS^k_i(Q_i', Q_{-i}) \supseteq R_i \setminus BOS^k_i(Q_i, Q_{-i}).$$  \hspace{1cm} (11)

Assume that there exists $Q_{-i}^*$ such that

$$BOS^k_i(Q_i, Q_{-i}^*) = P_i \setminus BOS^k_i(Q_i', Q_{-i}^*).$$  \hspace{1cm} (12)

This implies that $s^Q$ is acceptable as, by construction, $Q_i'$ contains no unacceptable schools.

Let $r_Q^s$ be the round of $BOS^k$ at which $t_i$ is assigned to $s^Q$ in $BOS^k_i$ given profile $(Q_i, Q_{-i}^*)$. Let $r^s_{Q'}$ be the rank of school $s^Q$ in strategy $Q_i'$. If $t_i$ is not assigned a school before round $r^s_{Q'}$ of $BOS^k$ given profile $(Q_i', Q_{-i}^*)$, then $t_i$ applies to $s^Q$ at round $r^s_{Q'}$. By construction, $t_i$ ranks the acceptable school $s^Q$ at a weakly lower rank in $Q_i'$ than in $Q_i$, which implies that $r^s_{Q'} \leq r^s_Q$.

Now, since by definition $BOS^k_i(Q_i, Q_{-i}^*) = s^Q$, the set of $t_j \neq t_i$ who apply to $s^Q$ before round $r_Q^s$, together with the set of $t_j \neq t_i$ who apply to $s^Q$ in round $r^s_{Q'}$ and have higher priority than $t_i$ at $s^Q$, has less than $q_{\text{QO}}$ students. But then, the set of $t_j \neq t_i$ who apply to $s^Q$ before round $r^s_Q < r_Q$, together with the set of $t_j \neq t_i$ who apply to $s^Q$ in round $r^s_{Q'}$ and have higher priority than $t_i$ at $s^Q$ also has less than $q_{\text{QO}}$ students. Therefore, $t_i$ is assigned a school at a round $r'' \leq r^s_{Q'}$ of the algorithm $BOS^k$ for profile $(Q_i', Q_{-i}^*)$, or in other words

$$BOS^k_i(Q_i', Q_{-i}^*) \supseteq Q_i' \setminus s^Q.$$  \hspace{1cm} (13)

Now, by construction of $Q_i'$, for all ranks $h \in \{1, \ldots, r'\}$, the school $Q_i'(h)$ satisfies

$$Q_i'(h) \cap Q_i' \cap s^Q,$$  \hspace{1cm} (13)
and is such that either

(i) \( Q'_i(h) = Q_i(h) \), or

(ii) \( Q'_i(h) \ R_i \ s^Q \).

In the construction, (ii) corresponds to the cases in which either \( Q_i(h) \not\in R_i \), or \( Q_i(h) \in R_i \) but \( Q_i(h) = Q'_i(h) \) for some \( h < h \). In these cases, the construction prescribes to set \( Q'_i(h) \) to the most preferred school according to \( R_i \), which is not yet ranked in \( Q'_i(h) \), for some \( h < h \). Because we only look at \( h \) such that (13) holds, \( s^Q \) has not yet been ranked, and hence, (ii) must hold.

Now, let us compare the effect of ranking \( Q_i \) with the effect of ranking \( Q'_i \) round by round in \( BOS^k \), for rounds \( r \leq r' \) (when the students \( t_j \not= t_i \) rank \( Q_{-i}^* \)). Because \( r'' \leq r_{Q'}^Q \leq r_Q \), \( t_i \) is rejected from the school she applies to in every round \( r < r'' \) when ranking \( Q_i \). Thus at each round \( r < r'' \), either

1. (i) holds and \( t_i \) is also rejected at round \( r \) when ranking \( Q'_i \),

2. (i) does not hold and (ii) holds, that is

\[
Q'_i(h) \ P_i \ s^Q
\]  

(14)

Then either

(a) \( t_i \) is rejected from \( Q'_i(h) \) at round \( r \), or

(b) \( t_i \) is accepted at \( Q'_i(h) \) at round \( r \).

But given (14), 2.(b) clearly contradicts (12). Thus \( t_i \) must be rejected at every round \( r < r_{Q'}^Q \) of \( BOS^k \) when ranking \( Q'_i \).

Now, this implies that \( BOS^k \) will move on to round \( r' \), implying \( r'' = r' \).

But by (13), this means

\[
BOS^k_i(Q'_i, Q_{-i}^*) = s^Q,
\]

again contradicting (12). Hence, (11) must hold.

In order to prove that the constructed \( Q'_i \) dominates \( Q_i \), there remains to show that there exists \( Q^*_{-i} \) such that

\[
BOS^k_i(Q'_i, Q^*_{-i}) \ P_i \ BOS^k_i(Q_i, Q^*_{-i}).
\]

By the definition of \( Range(Q_i) \), for every school \( s \in Range(Q_i) \), there exists \( Q^*_{-i} \) such that

\[
BOS^k_i(Q_i, Q^*_{-i}) = s.
\]

This is also true for any unacceptable school \( s' \in Range(Q_i) \). By assumption, there exists an unacceptable \( s' \in Range(Q_i) \). Since \( Q'_i \) contains only acceptable schools, we have that either

---

18 Although the schools ranked before rank \( r \) may differ in \( Q_i \) and \( Q'_i \), the fact that \( t_i \) was not assigned to any school yet means that the set of students assigned before round \( r \) is the same for \( (Q_i, Q^*_{-i}) \) and for \( (Q'_i, Q^*_{-i}) \) and that those students are assigned to the same schools. Hence, the remaining students in round \( r \) are the same and apply to the same schools for \( (Q_i, Q^*_{-i}) \) and for \( (Q'_i, Q^*_{-i}) \).
• $BOS^k_i(Q'_i, Q'^*_i)$ is acceptable, or
• $BOS^k_i(Q'_i, Q'^*_i) = t_i$.

In both cases we have $BOS^k_i(Q'_i, Q'^*_i)$, $P_i$, $s'$ and therefore $Q'^*_i$ qualifies for $Q'^*_i$.

Lemma 6

As the proof of sufficiency is obvious, we only prove necessity. We prove the contrapositive. Assume that neither (i) nor (ii) are true. Consider any sub-profile $Q'^*_i$ constructed as follows

• Take any set of $q_{Q'_i(1)}$ students $t_j \neq t_i$ among the students with higher priority at $Q'_i(1)$ than $t_i$, and let $Q'^*_j(1) := Q_i(1)$.

... 

• For any $\ell \in \{2, \ldots, \#Q'_i\}$ take $q_{Q'_i(\ell)}$ students $t_k$ whose reported preferences have not been constrained yet and let $Q'^*_k(1) := Q_i(\ell)$.

Because (i) is false, there are at least $q_{Q'_i(1)}$ students in $T$ with higher priority at school $Q'_i(1)$ than $t_i$. Because (ii) is false, any oversupplied set of schools contains more than $k$ schools. Therefore, \{Q_i(1), \ldots, Q_i(\#Q_i)\} is not an oversupplied set of schools. Hence, there are enough students to construct the sub-profile $Q'^*_i$, described above and $Q'^*_i$ is well-defined.

Because $Q_i(1)$ is not safe-if-favorite for $t_i$ we have $BOS^k_i(Q_i, Q'^*_i) \neq Q_i(1)$ by construction of $Q'^*_i$. By construction again, for every $s \in Q_i$ with $s_j \neq Q'_i(1)$, there are at least $q_{j}$ students who apply to $s_j$ in the first round of $BOS^k_i$. Therefore, $BOS^k_i(Q_i, Q'^*_i) \neq s_j$ and we have $BOS^k_i(Q_i, Q'^*_i) = t_i$, showing that $Q_i$ is not a safe strategy.

Lemma 7

The proof is by contradiction. Assume that $Q_i$ dominates $Q'_i$ and that $Q_i$ is unsafe. If $Q'_i$ is safe, it is obvious that the unsafe strategy $Q_i$ does not dominate the safe $Q'_i$ given that $Q'_i$ contains only acceptable schools. Therefore, we focus on the case in which $Q'_i$ is unsafe too.

We show that $Q_i$ does not dominate $Q'_i$. Again, this is trivially true if $Q_i = Q'_i$ and we therefore focus on the case $Q_i \neq Q'_i$.

Let $r$ be the lowest rank for which $Q_i(r) \neq Q'_i(r)$. There are two cases.

Case 1: $r > \#Q'_i$

This case implies that strategy $Q'_i$ is the truncation of $Q_i$ after rank $\#Q'_i$. Therefore, we have $\#Q'_i = \#R_i < k$. As a consequence, $Q_i(r)$ is unacceptable because all acceptable schools are ranked in $Q_i$ before rank $r$. Because $Q_i$ is
unsafe, there exists $Q^*_{-i}$ such that $BOS^k_i(Q_i, Q^*_{-i}) = Q_i(r)$; that is, $t_i$ is assigned to an unacceptable school.\footnote{This is proven formally in Lemma 8, see below. The construction of $Q^*_{-i}$ follows a procedure similar to the one introduced in the proof of Lemma 6.} As all schools ranked in $Q'_i$ are acceptable, $t_i$ strictly prefers her assignment when ranking $Q'_i$ and other students rank $Q^*_{-i}$. This shows that strategy $Q_i$ does not dominate $Q'_i$.

**Case 2:** $r \leq \#Q'_i$.

We construct $Q^*_{-i}$ such that

$$BOS^k_i(Q'_i, Q^*_{-i}) \cdot P_i \cdot BOS^k_i(Q_i, Q^*_{-i}) = t_i,$$

that is $t_i$ is assigned to an acceptable school when playing $Q'_i$ and unassigned when playing $Q_i$. We consider two constructions for two different cases.

**Construction 1** : $Q'_i(r) \in Q_i$.

- Take the $q_{Q_i(1)}$ students $t_j \neq t_i$ with the highest priority at school $Q_i(1)$ and let $Q'_j : Q_i(1)$,
- For all $s \in Q_i$ with $s \neq Q_i(1)$ and $s \neq Q'_i(r)$, take $q_s$ students $t_u$ whose reported preferences are not yet constrained and let $Q^*_u : s$,
- Take $q_{Q_i(r)} - 1$ students $t_v$ whose reported preferences are not yet constrained and let $Q^*_v : Q'_i(r)$,
- Take a student $t_g$ whose reported preferences are not yet constrained. If $t_i F_{Q'_i(r)} t_g$ then $Q^*_g$ is the truncation of $Q'_i$ after school $Q'_i(r)$, else it is the truncation of $Q_i$ after school $Q_i(r)$ with in addition $Q_g(r + 1)^* := Q'_g(r)$.
- Students whose preference is not specified yet do not rank school $Q'_i(r)$.

We show that $Q^*_{-i}$ is well-defined. First, $Q_i$ is unsafe and hence, $Q_i(1)$ is not safe-if-favorite. As a result, there are enough students $t_j$ in $T$ for the first step of the construction. Second, there are enough students to construct $Q^*_{-i}$ because the number of students whose preference is constrained (including student $t_i$) is equal to the sum of the seats available at the schools ranked in the unsafe $Q_i$.\footnote{Recall that $Q_i(r) \in Q_i$.} As $Q_i$ is unsafe, the sum of the seats available at these schools is no greater than $n$ by Lemma 6. Therefore, $Q^*_i$ can be constructed.

By construction, $BOS^k_i(Q_i, Q^*_{-i}) = t_i$, as all the seats at all the schools ranked in $Q_i$ are allocated at round 1 of the algorithm to other students than $t_i$, except for one seat at school $Q'_i(r)$ if $r \neq 1$. This last seat is allocated to $t_i$ at round $r$ for strategy $Q'_i$ and is allocated to $t_g$ at round $r$ or $r + 1$ for strategy $Q_i$.

**Construction 2** : $Q'_i(r) \notin Q_i$.

The construction of $Q^*_{-i}$ is almost identical. The only difference is that no student $t_v$ is constrained to rank $Q^*_v : Q'_i(r)$ and that $t_g$’s preferences are not constrained.

### C.2 Characterizing dominant strategies

**Proposition 3**

The sufficiency of the two conditions is obvious.
For the case in which \( \#R_i = 1 \) and \( R_i(1) \) is not safe-if-favorite, the necessity of condition (ii) is a corollary of the fact that a single class of strategies \( Q_i = R_i \) qualify as undominated strategies, as proven in Proposition 4 (see below).

To complete the proof, let us show that condition (i) is necessary when \( \#R_i \geq 2 \). We consider two cases.

**Case 1:** \( R_i(1) \) is safe-if-favorite but \( Q_i(1) \neq R_i(1) \).

By Proposition 4, such \( Q_i \) is not an undominated strategy in \( BOS^k \). Hence, \( Q_i \) is not a dominant strategy either.

**Case 2:** \( R_i(1) \) is not safe-if-favorite.

We prove that there exists no dominant strategies by showing the existence of \( Q'_i \) and \( Q''_i \) such that

- \( Q'_i \) and \( Q''_i \) are undominated strategy in \( BOS^k \) and
- \( Q'_i \) and \( Q''_i \) are not equivalent strategies.

Let \( Q'_i \) be such that \( Q'_i(1) := R_i(1) \) and \( Q'_i \) contains \( \min\{k, \#R_i\} \) acceptable schools. Strategy \( Q'_i \) is an undominated strategy by Proposition 4. Let \( Q''_i \) be such that \( Q''_i(1) := R_i(2) \) and \( Q''_i \) contains \( \min\{k, \#R_i\} \) acceptable schools. Strategy \( Q''_i \) is an undominated strategy by Proposition 4 because \( \#R_i \geq 2 \) and \( R_i(1) \) is not safe-if-favorite. Observe that this is true whether or not \( R_i(2) \) is safe-if-favorite.

There exists \( Q^*_{-i} \) – for example \( Q^*_{j} \) contains no school for all \( t_j \neq t_i \) – for which \( Q'_i \) and \( Q''_i \) yield different assignments and they are hence not equivalent.

**Corollary 2**

We first show that \( t_i \) is not part of a blocking pair. By Proposition 3, if \( Q_i \) is a dominant strategy, then two cases can arise:

**Case 1:** \( R_i(1) \) is safe-if-favorite and \( Q_i(1) = R_i(1) \).

This case is such that \( BOS^k_i(Q) = R_i(1) \). Hence, there exists no \( s \) with \( s P_i BOS^k_i(Q) \) and \( t_i \) cannot be part of a blocking pair.

**Case 2:** \( \#R_i = 1 \) and \( Q_i = R_i \).

If \( BOS^k_i(Q) = R_i(1) \), then \( t_i \) cannot participate to a blocking pair for the reason explained in Case 1. If on the other hand \( BOS^k_i(Q) = t_i \), then student \( t_i \) was rejected from \( R_i(1) \) in the first round of \( BOS^k \) for profile \( Q \). This implies that \( q_{R_i(1)} \) students with higher priority than \( t_i \) at \( R_i(1) \) are assigned to \( R_i(1) \) after the first round of the algorithm. The same students are assigned to \( R_i(1) \) at the end of the algorithm. Therefore \( t_i \) cannot participate in a blocking pair as she only finds \( R_i(1) \) acceptable.

It is easy to see that if \( t_i \) is assigned to a unacceptable school or prefers a school \( s \) with an available seat to her assignment, strategy \( Q_i \) cannot be dominant. Respectively, ranking no acceptable school and ranking \( s \) as the only acceptable school dominates \( Q_i \).
C.3 Characterizing undominated strategies

Proposition 4

Sufficiency.

Case (i)

If $Q_i(1)$ is the favorite acceptable safe-if-favorite, then $Q_i$ is a safe strategy by Lemma 6. By the definition of an unsafe strategy, only a safe strategy $Q'_i$ can dominate the safe $Q_i$ that guarantees assignment in the acceptable school $Q_i(1)$. By Lemma 6, $Q'_i$ is safe if and only if $Q'_i(1)$ is safe-if-favorite (given the short-supply assumption). As $Q_i(1)$ is the favorite acceptable safe-if-favorite, any strategy $Q'_i$ that dominates $Q_i$ must be such that $Q'_i(1) = Q_i(1)$. As $Q_i(1)$ is safe-if-favorite, the two strategies lead to the same assignment for $t_i$ whatever the strategies reported by other students. Hence $Q_i$ and $Q'_i$ are equivalent and $Q'_i$ cannot dominate $Q_i$.

Case (ii)

If $Q_i(1)$ is not safe-if-favorite, then $Q_i$ is unsafe by Lemma 6. By Lemma 7, if $Q_i$ is dominated, then $Q_i$ is dominated by a safe strategy $Q'_i$. Again, the safe strategy $Q'_i$ must be such that $Q'_i(1)$ is safe-if-favorite (Lemma 6).

Now, by assumption, there exists a school $s' \in Q_i$ that is preferred to the favorite acceptable safe-if-favorite. This guarantees that $Q'_i$ does not dominate $Q_i$. Indeed, Lemma 8 (see below) shows that for any school $s$ ranked in an unsafe strategy, there exists $Q^*_{-i}$ such that

$$BOS^k_i(Q_i, Q^*_{-i}) = s.$$  

In particular, there exists $Q^*_{-i}$ such that

$$BOS^k_i(Q_i, Q^*_{-i}) = s'.$$

As $Q'_i(1)$ is safe-if-favorite we have $BOS^k_i(Q'_i, Q^*_{-i}) = Q'_i(1)$. By assumption we also have $s' \in Q'_i(1)$, which shows that strategy $Q'_i$ does not dominate $Q_i$.

Necessity.

Case (i)

If $Q_i(1)$ is safe-if-favorite but not the favorite acceptable safe-if-favorite, it is clearly dominated by $Q'_i$ for which $Q'_i(1)$ is the favorite acceptable safe-if-favorite.

Case (ii)

Now, suppose $Q_i(1)$ is not safe-if-favorite. Assume first that $Q_i$ contains no school preferred to the favorite safe-if-favorite. Then it is again dominated by any $Q'_i$ for which $Q'_i(1)$ is the favorite safe-if-favorite. Assume now that $Q_i$ contains less than $\min\{k, \#R_i\}$ acceptable schools. Two cases can arise:

- $Q_i$ contains unacceptable schools.

  As $Q_i(1)$ is not safe-if-favorite, all schools in $Q_i$ belong to $\text{Range}(Q_i)$. By Lemma 5, $Q_i$ cannot be undominated strategy.

31
• $Q_i$ contains no unacceptable schools but less than $\min\{k, \#R_i\}$ acceptable schools.

In this case, there exists an acceptable school $s$ that is not ranked in $Q_i$. Strategy $Q'_i : Q_i s$ obtained by attaching $s$ at the end of $Q_i$ can be played in $BOS^k$ and dominates $Q_i$. By construction of $Q'_i$ we have that if

$$BOS^k_i(Q_i, Q^*_i) \neq BOS^k_i(Q'_i, Q^*_i),$$

then $BOS^k_i(Q_i, Q^*_i) = t_i$ and $BOS^k_i(Q'_i, Q^*_i) = s$. Since both strategies $Q_i$ and $Q'_i$ are unsafe, there exists such a $Q^*_i$ by Lemma 5.

**Lemma 8.** Let $Q_i$ be an unsafe strategy of $BOS^k$, where $k \in \mathbb{N}$. For any school $s \in Q_i$, there exists $Q^*_i$ such that

$$BOS^k_i(Q_i, Q^*_i) = s.$$

Proof. By definition of an unsafe strategy, there exists $Q^*_i$ such that $BOS^k_i(Q_i, Q^*_i) = t_i$.

Now consider $Q^*_i$ in which all $t_j$ assigned in $BOS^k_i(Q_i, Q^*_i)$ report

$$Q^*_j : BOS^k_i(Q_i, Q^*_i) \ t_j$$

and, for simplicity, students $t_h$ who are unassigned in $BOS^k_i(Q_i, Q^*_i)$ rank no schools at all in $Q^*_i$.\footnote{This is for simplicity only. By no means does the argument of the proof require that students be allowed to rank no schools. Other more realistic constructions of $Q^*_i$ would also do the job.} Clearly, we still have

$$BOS^k_i(Q_i, Q^*_i) = t_i$$

because the same set of students apply to $Q_i(1)$ in the first round (which implies that $t_i$ is still rejected from $Q_i(1)$ in the first round) and all the seats at all schools are filled in the first round.

Now construct $Q^*_1$ from $Q^*_i$ by changing only the reported profile of students $t_j \neq t_i$ who rank $Q_i(1)$, and make those students rank no schools at all. Then if $\ell = 1$, $BOS^k_i(Q_i, Q^*_i) = Q_i(1)$, as requested. Also, if $\ell > 1$, $t_i$ is still rejected from $Q_i(1)$ in the first round and all seats at all schools ranked before $Q_i(\ell)$ in $Q_i$ are filled in the first round. Therefore, we clearly have $BOS^k_i(Q_i, Q^*_i) = Q_i(1)$, the desired result. $\blacksquare$

**C.4 Comparing the extent of uncontroversial advice in $DA^k$ and $BOS^k$**

**Corollary 3**

**Part 1.** Every clean undominated strategy of $DA^k$ is an undominated strategy of $BOS^k$.

Clearly, if $R_i(1)$ is safe-if-favorite, strategies $Q_i$ with $Q_i(1) = R_i(1)$ are the only undominated strategies in both $BOS^k$ and $DA^k$. Thus we only need to prove the corollary for the case in which $R_i(1)$ is not safe-if-favorite.

By Proposition 2, there are two subcases.
Condition (i) in Proposition 2.
In this case, any clean undominated strategy $Q_i$ in $DA^k$ has $Q_i(1) = R_i(1)$ and ranks $\min\{k, \#R_i\}$ acceptable schools. Thus by Proposition 4(ii), $Q_i$ is also an undominated strategy in $BOS^k$ ($t_i$ prefers $Q_i(1)$ to all her acceptable safe-if-favorite schools).

Condition (ii) in Proposition 2.
Again, let $Q_i$ be any clean undominated strategy satisfying the conditions in of Proposition 2(ii). Note that any safe-if-favorite school $s^*$ is a minimal safe set in $DA^k$. Thus, by the condition in of Proposition 2(ii), for all safe-if-favorite school $s^*$ in $BOS^k$, there is a school $s \in Q_i$ with $s \neq s^*$ such that $s P_i s^*$. But then, by of Proposition 4(ii), $Q_i$ is an undominated strategy of $BOS^k$.

Part 2. Not every undominated strategy of $BOS^k$ is a clean undominated strategy of $DA^k$.
Simply consider Example 1. Strategy $Q_4$ is undominated in $BOS^3$ (by Proposition 4(ii)) but as we show in the example, $Q_4$ is dominated in $DA^3$. Similar counter-example are easily found for any $k \geq 2$.

C.5 Comparing the manipulability with respect to the criterion of Arribillaga and Massó (2015)
Given preferences $R_i$, let $R_i^k$ denote a truthful strategy in a school choice mechanism with constraint $k$. Formally, $R_i^k : R_i(1) \ldots R_i(k) t_i \ldots$.

Lemma 9. For each preferences $R_i$ at which $t_i$ has a dominant strategy in $BOS^k$, this dominant strategy is equivalent in $BOS^k$ to the truthful strategy $R_i^k$.

Proof. By Proposition 3, there are two cases in which $t_i$ has a dominant strategy in $BOS^k$.

Case (i) : $\#R_i(1)$ is safe-if-favorite.
Then by Proposition 3(i), any dominant strategy $Q_i$ is such that $Q_i(1) = R_i(1)$.
Such $Q_i$ is equivalent to $R_i^k$ in $BOS^k$ because $Q_i(1) = R_i^k(1) = R_i(1)$ and $R_i(1)$ is safe-if-favorite.

Case (ii) : $\#R_i = 1$.
Then by Proposition 3(ii), the dominant strategy $Q_i$ is such that $Q_i = R_i$.
Such $Q_i$ is equivalent to $R_i^k$ in $BOS^k$ given that $R_i^k = R_i$ when $\#R_i = 1$.

Lemma 10. For any preferences $R_i$ for which $t_i$ has a dominant strategy in $DA^k$, this dominant strategy is equivalent in $DA^k$ to the truthful strategy $R_i^k$.

Proof. By Proposition 1, there are two cases in which $t_i$ has a dominant strategy in $DA^k$.

Case (a) : $\#R_i \leq k$.
Then, the dominant strategy $Q_i$ is such that $Q_i = R_i$ by Proposition 1(i). Such $Q_i$ is equivalent to $R_i^k$ in $DA^k$ given that $R_i^k = R_i$ when $\#R_i \leq k$. 

33
Case (b) : for some $q \leq \min\{k, \#R_i\}$, the $q$ most preferred schools in $R_i$ form a safe set.

Then, any dominant strategy $Q_i$ is such that this safe set is ranked first in $Q_i$ and there is no switch among those $q$ schools in $Q_i$ by Proposition 1(ii). Such $Q_i$ is equivalent to $R_i^k$ in $DA^k$ given that $Q_i(\ell) = R_i^k(\ell)$ for all $\ell \in \{1, \ldots, q\}$ and these $q$ schools form a safe set in $DA^k$.

\[\blacksquare\]

**Corollary 4**

**Part 1.** For each $R_i$ at which $t_i$ has a truthful dominant strategy in $BOS^k$, $t_i$ also has a truthful dominant strategy in $DA^k$.

By Lemma 9 and 10, it is sufficient to prove that for each $R_i$ at which $t_i$ has a dominant strategy in $BOS^k$, $t_i$ has a dominant strategy in $DA^k$.

By Proposition 3, there are two cases in which $t_i$ has a dominant strategy in $BOS^k$.

- **Case (a) :** $R_i(1)$ is safe-if-favorite.

  This implies that school $R_i(1)$ forms a safe set in $DA^k$. Hence, there exists $q = 1 \leq k$ such that the $q$ most preferred schools in $R_i$ form a safe set $DA^k$.

  Therefore, $t_i$ has a dominant strategy in $DA^k$ by Proposition 1(ii).

- **Case (b) :** $\#R_i = 1$.

  In this case, $t_i$ has a dominant strategy in $DA^k$ by Proposition 1(i).

**Part 2.** For all $k > 1$, there exist $R_i$, $F$ and $q$ such that $t_i$ has a truthful dominant strategy in $DA^k$ but not in $BOS^k$.

By Lemma 9 and 10, it is sufficient to prove that there exists $R_i$, $F$ and $q$ such that $t_i$ has a dominant strategy in $DA^k$ but not in $BOS^k$.

Take for example any preferences $R_i$ featuring $k$ acceptable schools for which $R_i(1)$ is not safe-if-favorite. Such preferences always exist because of the short-supply assumption. The strategy $Q_i = R_i$ is dominant in $DA^k$ by Proposition 1(i). Strategy $Q_i$ is not dominant in $BOS^k$ since $R_i(1)$ is not safe-if-favorite (in violation of condition (i) in Proposition 3) and $\#R_i = \#Q_i > 1$ for all $k > 2$ (in violation of condition (i) in Proposition 3).

**Corollary 5**

**Part 1.** For each $R_i$ at which $t_i$ has a truthful dominant strategy in $DA^k$, $t_i$ has a truthful dominant strategy in $DA^{k+1}$.

By Lemma 9 and 10, it is sufficient to show that for each $R_i$ at which $t_i$ has a dominant strategy in $DA^k$, $t_i$ has a dominant strategy in $DA^{k+1}$.

By Proposition 1, there are two cases for which $t_i$ has a dominant strategy in $DA^k$.

- **Case (i) :** $\#R_i \leq k$.

  Then, $\#R_i \leq k + 1$ and $t_i$ has a dominant strategy in $DA^{k+1}$.  
34
Case (ii) : for some $q \leq \min\{k, \#R_i\}$, the $q$ most preferred schools in $R_i$ form a safe set. Then, we have that $q \leq \min\{k + 1, \#R_i\}$ and the $q$ most preferred schools in $R_i$ form a safe set in $DA^{k+1}$. Therefore, $t_i$ has a dominant strategy in $DA^{k+1}$.

**Part 2.** For all $k < m$, there exist $R_i$, $F$ and $q$ such that $t_i$ has a truthful dominant strategy in $DA^{k+1}$ but not in $DA^k$.

By Lemma 9 and 10, it is sufficient to show that if there exist $R_i$, $F$ and $q$ such that $t_i$ has a dominant strategy in $DA^{k+1}$ but not in $DA^k$.

Take for example any preference $R_i$ featuring $k + 1$ acceptable schools such that $R_i$ contains no safe set. Such preferences always exist by the short-supply assumption. Any strategy $Q_i = R_i$ is dominant in $DA^{k+1}$ by Proposition 1(i). Strategy $Q_i$ is not well-defined in $DA^k$ since $\#Q_i > 1$. Furthermore, no strategy $Q_i'$ is dominant in $Q_i$ as $\#R_i > k$ (in violation of condition (i) in Proposition 1) and $R_i$ contains no safe set (in violation of condition (ii) in Proposition 1).

**Corollary 6**

For each $R_i$ at which $t_i$ has a truthful dominant strategy in $BOS^k$, $t_i$ has a truthful dominant strategy in $BOS^q$ for all $k,q \geq 1$.

By Lemma 9 and 10, it is sufficient to prove that for each $R_i$ at which $t_i$ has a dominant strategy in $BOS^k$, $t_i$ has a dominant strategy in $BOS^q$ for all $k,q \geq 1$.

By Proposition 3, there are two cases for which $t_i$ has a dominant strategy in $BOS^k$. The first case arises when $R_i(1)$ is safe-if-favorite. This implies that $t_i$ has a dominant strategy in $BOS^q$. The second case arises when $\#R_i = 1$. Therefore, $t_i$ has a dominant strategy in $BOS^q$.

**References**


